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Generalized Macdonald–Ruijsenaars systems [☆]

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Abstract

We consider the polynomial representation of Double Affine Hecke Algebras (DAHAs) and construct its submodules as ideals of functions vanishing on the special collections of affine planes. This generalizes certain results of Kasatani in types A_n , (C_n^\vee, C_n) . We obtain commutative algebras of difference operators given by the action of invariant combinations of Cherednik–Dunkl operators in the corresponding quotient modules of the polynomial representation. This gives known and new generalized Macdonald–Ruijsenaars systems. Thus in the cases of DAHAs of types A_n and (C_n^\vee, C_n) we derive Chalykh–Sergeev–Veselov operators and a generalization of the Koornwinder operator respectively, together with complete sets of quantum integrals in the explicit form.

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Keywords: Double Affine Hecke Algebra; Polynomial representation; Wheel condition; Macdonald operator; Integrable system

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1. Introduction

The main goal of this paper is to obtain commutative algebras of difference operators containing generalizations of Macdonald–Ruijsenaars operators using special representations of Double Affine Hecke Algebras (DAHAs). In the work [27] the Macdonald–Ruijsenaars operator related to the root system $R = A_{N-1}$ appeared as the relativistic version of quantum Calogero–Moser system, this operator has the form

$$H = \sum_{i=1}^N \prod_{\substack{j=1 \\ j \neq i}}^N \frac{e^{x_i} - e^{x_j + \eta}}{e^{x_i} - e^{x_j}} \mathcal{T}_{x_i}^{\hbar}, \quad (1.1)$$

where $\mathcal{T}_{x_i}^{\hbar} = e^{\hbar \partial_{x_i}}$ is the shift operator and η, \hbar are parameters. Macdonald derived and considered operator (1.1) and its generalizations for any root system and (quasi-)minuscule coweight in the context of multi-variable orthogonal polynomials [25]. Commutative algebras of difference operators containing the operators constructed by Macdonald were realized by Cherednik with the help of his DAHA and its polynomial representation [5,8]. These algebras are centralizers of the Macdonald–Ruijsenaars operators in the algebras of Weyl-invariant difference operators [22]. The whole area of DAHA is a very rich mathematical subject with deep connections and interactions with combinatorics, theory of (quantum) symmetric spaces, representation theory, noncommutative geometry [9].

The first generalization of the Macdonald–Ruijsenaars operator which is not directly associated to a root system was found by Chalykh [2]. The operator is a difference version of the Hamiltonian which describes interaction of the system of Calogero–Moser particles of two possible masses [4]. The approach of [2] (see also [3]) is based on the multidimensional Baker–Akhiezer functions, in this case all but one particles have equal masses. The generalization of Chalykh’s operator to the system of $N_1 + N_2$ particles of two types was introduced by Sergeev and Veselov [29], the corresponding operator has the form

$$\begin{aligned}
H_{N_1, N_2} = & \sum_{i=1}^{N_1} \prod_{\substack{j=1 \\ j \neq i}}^{N_1} \frac{e^{x_i} - e^{x_j + \eta}}{e^{x_i} - e^{x_j}} \prod_{\ell=1}^{N_2} \frac{e^{x_i} - e^{y_\ell + \hbar}}{e^{x_i} - e^{y_\ell}} \mathcal{T}_{x_i}^\hbar \\
& + \frac{e^\hbar - 1}{e^\eta - 1} \sum_{\ell=1}^{N_2} \prod_{i=1}^{N_1} \frac{e^{y_\ell} - e^{x_i + \eta}}{e^{y_\ell} - e^{x_i}} \prod_{\substack{m=1 \\ m \neq \ell}}^{N_2} \frac{e^{y_\ell} - e^{y_m + \hbar}}{e^{y_\ell} - e^{y_m}} \mathcal{T}_{y_\ell}^\eta.
\end{aligned} \tag{1.2}$$

In the work [29] the operator (1.2) was obtained from the action of the Macdonald–Ruijsenaars operator (1.1) extended to the space of symmetric functions in the infinite number of variables with subsequent restriction of the operator (1.1) to the special discriminant subvariety. Further rational generalized Macdonald–Ruijsenaars operators were derived in [12,30] starting from the generalized Calogero–Moser systems and using the bispectrality.

In the present work we develop a uniform approach to the generalized Macdonald–Ruijsenaars systems based on the special representations of DAHAs. DAHA can be associated to any affine root system [25]. The main cases are DAHA of type \widehat{R} and of type \widehat{R}^\vee where R is a reduced irreducible root system, \widehat{R} is its affinization and \widehat{R}^\vee denotes the dual affine root system (see Cherednik’s papers [7] and [5] respectively), and DAHA of type (C_n^\vee, C_n) (see [26,28]). The original Cherednik’s construction [5,7] allowed to obtain Macdonald–Ruijsenaars systems for any root system R from the polynomial representation of the corresponding DAHA. The difference operators from the commutative algebra are obtained by the averaging of the Cherednik–Dunkl operators over the Weyl group with subsequent restriction of the result to the space of invariant polynomials. In [26] Noumi generalized the construction for the (C_n^\vee, C_n) case which allowed to obtain commutative algebra containing the Koornwinder operator [21]. In order to obtain generalized Macdonald–Ruijsenaars systems we first specialize DAHA-invariant ideals in the polynomial representation, these ideals consist of functions vanishing on certain collections of planes. Then the action of the invariant combinations of Cherednik–Dunkl operators is defined on the functions on these planes. Upon restriction to Weyl group invariants we obtain commutative algebras of difference operators. In this way we recover operator (1.2) (under assumption that $\hbar/\eta \in \mathbb{Z}$) for the case of DAHA of type A and derive further known and new generalized Macdonald–Ruijsenaars operators starting from other root systems. Similar approach to the generalized Calogero–Moser systems was developed by one of the authors in [13]. It is based on finding special ideals invariant under the rational Cherednik algebras with subsequent restriction of invariant combinations of Dunkl operators thus generalizing Heckman’s construction of the usual Calogero–Moser systems [16].

The idea to describe submodules of the polynomial representations of DAHAs by vanishing conditions was very successfully used by Kasatani in [18,19] (see also discussion in the introduction of [10] where the polynomial representation is studied from another perspective). Thus in the work [18] a filtration of the polynomial representation in type A by the ideals given by vanishing (“wheel”) conditions was introduced so that it provides composition factors of the polynomial module (see also [11]). The paper [19] deals with the polynomial representation of DAHA associated to the affine root system (C_n^\vee, C_n) . Ideals in the ring of symmetric polynomials invariant under the Macdonald–Ruijsenaars operator (1.1) were earlier defined by wheel conditions in [15].

Submodules in the polynomial representations exist at special relations between parameters of the DAHAs. Therefore the generalized Macdonald–Ruijsenaars operators which we obtain carry these restrictions. This is different from the type A construction of [29] where the authors work in the space of symmetric functions in infinitely many variables and consider some invariant ideals

in that algebra. Nevertheless, for the classical series of root systems one can consider polynomial representations for different ranks of root systems and the respective invariant submodules so that the resulting operators coincide. That is one obtains operators at some integral restriction on the parameters but the corresponding integral values can be chosen arbitrarily and the operators do not depend on these values. This allows to omit these restrictions and to continue the commuting families of the resulting operators into the general parameters. In type A_n this leads to the operator (1.2) and its quantum integrals. In type (C_n^\vee, C_n) this leads to the generalizations of Koornwinder and van Diejen operators.

The structure of the paper is as follows. In Section 2 we recall Cherednik's construction of the commutative algebra of difference operators using DAHA of type \widehat{R} . In Section 3 we introduce arrangements of planes \mathcal{D} corresponding to subsets S_0 of the set of simple roots of the root system R and we define ideals of functions $I_{\mathcal{D}}$ vanishing on these planes. Our first main result (Theorem 3.3) states that for the special geometry of S_0 and for the related choice of parameters for DAHA the ideals $I_{\mathcal{D}}$ are DAHA-invariant. The arising arrangements \mathcal{D} are shifted versions of the linear arrangements appearing in the study of invariant ideals for the rational Cherednik algebras [13]. In Section 3.4 we construct further invariant ideals given by vanishing conditions where we allow parallel planes. This generalizes construction of such ideals for DAHA for classical root systems from [18,19].

In Section 4 we present our next main result (Theorem 4.1) which explains how commutative algebras of difference operators arise in the quotients of the polynomial representations by the above ideals $I_{\mathcal{D}}$. We show in Section 4.2 that these algebras contain complete sets of algebraically independent operators. We verify in Section 4.3 that in type A we obtain operators (1.2). We also construct explicitly a complete set of commuting operators for (1.2) (Theorem 4.5). We consider further examples illustrating the approach in Sections 4.4, 4.5. Thus we demonstrate that starting from the Macdonald operator of type B we obtain trigonometric version of the operators known from [30,12]. Subsystems in the exceptional root systems provide new families of commutative algebras, we demonstrate this by getting a restricted operator starting from the Macdonald operator for the root system $R = F_4$.

In Section 5 we extend our constructions for the (C_n^\vee, C_n) DAHA. We specify invariant ideals in the polynomial representation by imposing vanishing condition on suitable arrangements of planes (Theorem 5.2). These submodules correspond to the intersections of the submodules investigated in [19]. We introduce generalized Koornwinder operator and derive it as a restriction of the Koornwinder operator in Propositions 5.4, 5.5. This operator depends on six parameters like the original Koornwinder operator, and also it reflects that there are interacting particles of two types. Although the restriction construction imposes some integrality relation(s) on the parameters we show that the operator can be embedded into a large commutative algebra of difference operators for any values of the parameters. In Theorem 5.7 we find a complete set of quantum integrals in the explicit form. This generalizes van Diejen's quantum integrals for the Koornwinder operator [31].

In Section 6 we present our constructions for DAHA of type \widehat{R}^\vee [5]. The invariant ideals are specified in Theorem 6.1.

2. Macdonald–Ruijsenaars systems from DAHA

In this section we recall the construction of the Macdonald–Ruijsenaars systems from the Cherednik–Dunkl operators. We consider here DAHAs of type \widehat{R} (other cases will be considered in Sections 5, 6 and Appendix A). Following [7–9,20] we define DAHA for the \widehat{R} case,

then we define the Cherednik–Dunkl operators as elements of DAHA acting in the polynomial representation and describe the Macdonald–Ruijsenaars systems in these terms.

2.1. Double Affine Hecke Algebra

Let $R \subset \mathbb{R}^n \subset V = \mathbb{C}^n$ be an irreducible reduced root system of rank n . Let (\cdot, \cdot) be the standard bilinear symmetric non-degenerated form in V and let $\beta^\vee = 2\beta/(\beta, \beta)$. We have $(\alpha, \beta^\vee) \in \mathbb{Z}$ for all $\alpha, \beta \in R$. Consider the orthogonal reflections s_α corresponding to the roots $\alpha \in R$, they act on $x \in V$ as $s_\alpha(x) = x - (\alpha, x)\alpha^\vee$. The group W generated by the orthogonal reflections $s_\alpha, \alpha \in R$, is a Weyl group.

Let R_+ be a set of positive roots and $\Delta = \{\alpha_1, \dots, \alpha_n\} \subset R_+$ be the set of simple roots. The reflections $s_{\alpha_1}, \dots, s_{\alpha_n}$ generate the Weyl group W . Let us introduce lattices of coroots, of coweights and of weights:

$$Q^\vee = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i^\vee, \quad P^\vee = \bigoplus_{i=1}^n \mathbb{Z}b_i, \quad P = \bigoplus_{i=1}^n \mathbb{Z}\omega_i, \quad (2.1)$$

where b_i and ω_i are fundamental coweights and fundamental weights:

$$(\alpha_i, b_j) = \delta_{ij}, \quad (\omega_i, \alpha_j^\vee) = \delta_{ij}. \quad (2.2)$$

Introduce also the subsets of dominant and anti-dominant coweights:

$$P_+^\vee = \{\lambda \in P^\vee \mid (\alpha, \lambda) \geq 0, \forall \alpha \in R_+\}, \quad P_-^\vee = \{\lambda \in P^\vee \mid (\alpha, \lambda) \leq 0, \forall \alpha \in R_+\}.$$

Let \widehat{V} be the space of affine functionals on V . Each functional $\alpha \in \widehat{V}$ acts on $x \in V$ as $\alpha(x) = (\alpha^{(1)}, x) + \alpha^{(0)}$, where $\alpha^{(1)} \in V$ and $\alpha^{(0)} \in \mathbb{C}$, we write $\alpha = [\alpha^{(1)}, \alpha^{(0)}]$ in this case. The roots $\gamma \in R$ can be considered as elements of \widehat{V} acting as $\gamma(x) = (\gamma, x)$. Define the bilinear form on the space \widehat{V} by the formula $(\alpha, \beta) = (\alpha^{(1)}, \beta^{(1)})$, where $\alpha = [\alpha^{(1)}, \alpha^{(0)}]$, $\beta = [\beta^{(1)}, \beta^{(0)}]$, and $\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)} = [\frac{2\alpha^{(1)}}{(\alpha^{(1)}, \alpha^{(1)})}, \frac{2\alpha^{(0)}}{(\alpha^{(1)}, \alpha^{(1)})}]$ for any $\alpha = [\alpha^{(1)}, \alpha^{(0)}] \in \widehat{V}$.

The reflection s_α corresponding to $\alpha = [\alpha^{(1)}, \alpha^{(0)}] \in \widehat{V}$ is the following affine map: $s_\alpha(x) = x - \alpha(x)(\alpha^{(1)})^\vee$. It is the orthogonal reflection with respect to the hyperplane $H(\alpha) = \{x \in V \mid \alpha(x) = 0\}$. The action s_α on V induces the action on \widehat{V} given by the formula $s_\alpha(\beta) = \beta - (\beta, \alpha^\vee)\alpha$, where $\alpha, \beta \in \widehat{V}$.

Let $\hbar \in \mathbb{C}$ be a non-zero complex parameter. Introduce the functional $\alpha_0 = [-\theta, \hbar]$, where $\theta \in R$ is the maximal root for R with respect to the basis Δ . For $\lambda \in V$ denote by $\tau(\lambda)$ the shift operator $\tau(\lambda)x = x + \hbar\lambda, \forall x \in V$.

Affine Weyl group \widetilde{W} is generated by the reflections s_α , with $\alpha = [\alpha^{(1)}, \alpha^{(0)}]$ such that $\alpha^{(1)} \in R$ and $\alpha^{(0)} \in \hbar\mathbb{Z}$. The set $\widehat{R} = \{[\alpha^{(1)}, \hbar k] \mid \alpha^{(1)} \in R, k \in \mathbb{Z}\}$ is the affine root system corresponding to the root system R [23]. The group $\tau(Q^\vee)$ is a normal subgroup in \widetilde{W} and $\widetilde{W} = W \ltimes \tau(Q^\vee)$. The group \widetilde{W} is generated by the reflections s_0, s_1, \dots, s_n , where $s_i = s_{\alpha_i}$ for $i = 0, 1, \dots, n$, (see [17]). The subset $\widehat{\Delta} = \{\alpha_0, \alpha_1, \dots, \alpha_n\} \subset \widehat{R}$ is a basis of \widehat{V} called the set of affine simple roots. For $0 \leq i, j \leq n, i \neq j$ we denote by $m_{ij} = m_{ji}$ the order (finite or infinite) of the element $s_i s_j \in \widetilde{W}$.

Extended affine Weyl group is defined as $\widehat{W} = W \ltimes \tau(P^\vee)$ with the commutation relations $w\tau(\lambda) = \tau(w\lambda)w$, where $w \in W, \lambda \in P^\vee$. The action of the group \widehat{W} on V induces the following linear action on the space \widehat{V} :

$$w\tau(\lambda)(\alpha) = [w(\alpha^{(1)}), \alpha^{(0)} - \hbar(\alpha^{(1)}, \lambda)], \quad (2.3)$$

where $w \in W, \lambda \in P^\vee, \alpha = [\alpha^{(1)}, \alpha^{(0)}]$.

Let $\Pi = \{\pi \in \widehat{W} \mid \pi \widetilde{\Delta} = \widetilde{\Delta}\} \subset \widehat{W}$ be a subgroup of elements preserving the set of simple affine roots. Then $\widehat{W} = \Pi \ltimes \widetilde{W}$. The group Π is finite: $\Pi = \{\pi_r \mid r \in \mathcal{O}\}$, where $\mathcal{O} \subset \{0, 1, \dots, n\}$ and each element π_r is uniquely determined by the condition $\pi_r(\alpha_0) = \alpha_r$. They have the form $\pi_0 = 1$, $\pi_r = \tau(b_r)u_r^{-1}$ for $r \in \mathcal{O}'$, where $\mathcal{O}' = \mathcal{O} \setminus \{0\}$ and $u_r \in W$ are transformations preserving the set $\{-\theta, \alpha_1, \dots, \alpha_n\}$ and satisfying $u_r(\alpha_r) = -\theta$. The corresponding fundamental coweights b_r form the entire set of *minuscule coweights*, that is $\{b_r \mid r \in \mathcal{O}'\} = \{\lambda \in P^\vee \mid 0 \leq (\alpha, \lambda) \leq 1, \forall \alpha \in R_+\}$.

Let $\widehat{P} = \{\mu = [\mu^{(1)}, \mu^{(0)}] \mid \mu^{(1)} \in P, \mu^{(0)} \in \frac{\hbar}{\epsilon_0} \mathbb{Z}\} \subset \widehat{V}$, where ϵ_0 is an integer such that $(P, P^\vee) \subset \epsilon_0^{-1} \mathbb{Z}$. We consider the weight lattice P as a sublattice of \widehat{P} . Now we define DAHA corresponding to the affine system \widehat{R} . Let $t: \widehat{R} \rightarrow \mathbb{C} \setminus \{0\}$ be a \widehat{W} -invariant function on \widehat{R} , that is $t_\beta = t_{\hat{w}(\beta)}$, for all $\beta \in \widehat{R}$, $\hat{w} \in \widehat{W}$, where we denote $t_\beta = t(\beta)$. We also denote $t_i = t_{\alpha_i}$ for $i = 0, 1, \dots, n$, these values uniquely determine the function t . Let $q = e^{\hbar/2}$. We will assume that $\hbar \notin \pi i \mathbb{Q}$, so q is not a root of unity.

Definition 2.1. ([7], see also [20].) Double Affine Hecke Algebra (for \widehat{R} case) is defined as the algebra $\mathcal{H}_t = \mathcal{H}_{\hbar, t}^{\widehat{R}}$ generated by the elements T_0, T_1, \dots, T_n , by the set of pairwise commuting elements $\{X^\mu\}_{\mu \in \widehat{P}}$, and by the group $\Pi = \{\pi_r \mid r \in \mathcal{O}\}$ with the relations

$$X^\mu X^{\mu'} = X^{\mu+\mu'}, \quad \mu, \mu' \in \widehat{P}; \quad (2.4)$$

$$X^v = q^{2s} = e^{\hbar s}, \quad v = [0, \hbar s] \in \widehat{P}, \quad s \in \epsilon_0^{-1} \mathbb{Z}; \quad (2.5)$$

$$(T_i - t_i)(T_i + t_i^{-1}) = 0; \quad (2.6)$$

$$\underbrace{T_i T_j T_i \dots}_{m_{ij} \text{ factors}} = \underbrace{T_j T_i T_j \dots}_{m_{ij} \text{ factors}}, \quad \text{if } i \neq j \text{ and } m_{ij} < \infty; \quad (2.7)$$

$$\pi_r T_i \pi_r^{-1} = T_j, \quad \text{if } \pi_r(\alpha_i) = \alpha_j, \quad r \in \mathcal{O}; \quad (2.8)$$

$$T_i^{-1} X^\mu T_i^{-1} = X^{\mu - \alpha_i}, \quad \text{if } (\mu, \alpha_i^\vee) = 1, \quad \mu \in \widehat{P}; \quad (2.9)$$

$$T_i X^\mu = X^\mu T_i, \quad \text{if } (\mu, \alpha_i^\vee) = 0, \quad \mu \in \widehat{P}; \quad (2.10)$$

$$\pi_r X^\mu \pi_r^{-1} = X^{\pi_r(\mu)}, \quad r \in \mathcal{O}, \quad \mu \in \widehat{P}; \quad (2.11)$$

where $i, j = 0, 1, \dots, n$.

Note that the generators $T_0, T_1, \dots, T_n, X^\mu, \pi_r$ of the algebra \mathcal{H}_t are invertible elements. We define an element $T_{\hat{w}} \in \mathcal{H}_t$ for each element $\hat{w} \in \widehat{W}$ according to its reduced expression:

$$T_{\hat{w}} = \pi_r T_{i_1} \dots T_{i_l}, \quad \text{for } \hat{w} = \pi_r s_{i_1} \dots s_{i_l}, \quad (2.12)$$

where $l = l(\hat{w})$ is the length of $\hat{w} \in \widehat{W}$: $l(\hat{w}) = |\widehat{R}_- \cap \hat{w} \widehat{R}_+|$, where $\widehat{R}_+ = -\widehat{R}_- = R_+ \cup \{[\alpha, \hbar k] \mid \alpha \in R, k \in \mathbb{Z}_{>0}\}$. In particular, $T_{\pi_r} = \pi_r$ and $T_{s_i} = T_i$. For $\hat{w}, \hat{v} \in \widehat{W}$ one has the relation

$$T_{\hat{v}\hat{w}} = T_{\hat{v}} T_{\hat{w}}, \quad \text{if } l(\hat{v}\hat{w}) = l(\hat{v}) + l(\hat{w}). \quad (2.13)$$

Since for the fundamental coweights we have $l(\tau(b_i + b_j)) = l(\tau(b_i)) + l(\tau(b_j))$ it leads to $T_{\tau(b_i)} T_{\tau(b_j)} = T_{\tau(b_j)} T_{\tau(b_i)}$ for all $i, j = 1, \dots, n$. Let us define the commutative family of elements Y^λ for $\lambda \in P^\vee$:

$$Y^\lambda = \prod_{i=1}^n (T_{\tau(b_i)})^{m_i}, \quad \text{if } \lambda = \sum_{i=1}^n m_i b_i, \quad m_i \in \mathbb{Z}. \quad (2.14)$$

They span a subalgebra $\mathbb{C}[Y]$ of \mathcal{H}_t isomorphic to $\mathbb{C}[P^\vee]$ and they satisfy the relations (see [20])

$$Y^\lambda Y^{\lambda'} = Y^{\lambda+\lambda'}, \quad \lambda, \lambda' \in P^\vee; \quad (2.15)$$

$$T_i Y^\lambda - Y^{s_i \lambda} T_i = (t_i - t_i^{-1}) \frac{Y^\lambda - Y^{s_i \lambda}}{1 - Y^{-\alpha_i^\vee}}, \quad \lambda \in P^\vee, \quad (2.16)$$

where $i = 1, \dots, n$. These formulae imply that the generators T_1, \dots, T_n commute with the elements of the subalgebra $\mathbb{C}[Y]^W \subset \mathbb{C}[Y]$ consisting of the polynomials invariant under the usual action $w : Y^\lambda \mapsto Y^{w\lambda}$ of the Weyl group W .

Denote by $C^\infty(V)^{2\pi i Q^\vee}$ the space of smooth periodic functions $f : V \rightarrow \mathbb{C}$ with the lattice of periods $2\pi i Q^\vee$, that is $f(x + 2\pi i \alpha_j^\vee) = f(x)$ for $j = 1, \dots, n$. Let

$$\mathcal{F} = \{f \in C^\infty(V)^{2\pi i Q^\vee} \mid f \text{ is holomorphic in a neighbourhood of } Z\}, \quad (2.17)$$

where $Z := \{x \in V \mid \alpha(x) \in 2\pi i \mathbb{Z} \text{ for some } \alpha \in \widehat{R}\}$. The space \mathcal{F} is \widehat{W} -invariant with respect to the action

$$\widehat{w} f(x) = f(\widehat{w}^{-1} x), \quad \widehat{w} \in \widehat{W}, \quad f \in \mathcal{F}, \quad x \in V. \quad (2.18)$$

The elements of the algebra \mathcal{H}_t can be represented as operators acting in \mathcal{F} :

$$X^\mu = e^{\mu(x)}, \quad \pi_r = \pi_r, \quad T_i = \frac{t_i - t_i^{-1}}{1 - e^{-\alpha_i(x)}} + \frac{t_i^{-1} - t_i e^{-\alpha_i(x)}}{1 - e^{-\alpha_i(x)}} s_i, \quad (2.19)$$

where $\mu \in \widehat{P}$, $r \in \mathcal{O}$, $i = 0, 1, \dots, n$. In order to show that $T_i f \in \mathcal{F}$ for all $f \in \mathcal{F}$ and $i = 0, 1, \dots, n$ we first rearrange the formula for T_i into the form

$$T_i = t_i + \frac{t_i^{-1} - t_i e^{-\alpha_i(x)}}{1 - e^{-\alpha_i(x)}} (s_i - 1). \quad (2.20)$$

Then we note that the function $g(x) = f(x) - f(s_i x) = f(x) - f(x - \alpha_i(x)(\alpha_i^{(1)})^\vee)$ belongs to \mathcal{F} and vanishes whenever $\alpha_i(x) \in 2\pi i \mathbb{Z}$. Hence the function $\tilde{g}(x) = \frac{g(x)}{1 - e^{-\alpha_i(x)}}$ is holomorphic in a neighbourhood of Z and smooth everywhere. So that $\tilde{g} \in \mathcal{F}$ and the function

$$T_i f(x) = t_i f(x) + (t_i^{-1} - t_i e^{-\alpha_i(x)}) \tilde{g}(x) \quad (2.21)$$

also belongs to \mathcal{F} .

The formulae (2.19) define a faithful representation of DAHA \mathcal{H}_t on the space \mathcal{F} called *polynomial representation* (extended to smooth functions). The elements Y^λ in this representation are called *Cherednik–Dunkl operators*.

2.2. Macdonald–Ruijsenaars systems

Denote by \mathcal{F}^W the subspace of invariant functions $f \in \mathcal{F}$, that is $f(wx) = f(x)$ for all $w \in W$ and $x \in V$. Consider the subalgebra $\mathbb{C}[Y]^W \subset \mathbb{C}[Y] \subset \mathcal{H}_t$. It is spanned as a vector space by the elements

$$m_\lambda(Y) = \sum_{\nu \in W\lambda} Y^\nu = \frac{1}{|W_\lambda|} \sum_{w \in W} Y^{w\lambda}, \quad (2.22)$$

where $W\lambda = \{w\lambda \mid w \in W\}$ is the orbit of the coweight $\lambda \in P^\vee$ and $W_\lambda = \{w \in W \mid w\lambda = \lambda\}$ is the stabilizer of λ . We will normally be choosing $\lambda \in P_-^\vee$ for the operators $m_\lambda(Y)$.

In the representation (2.19) the elements $m_\lambda(Y)$ have the form

$$m_\lambda(Y) = \sum_{v \in P^\vee} \sum_{w \in W} g_{v,w}^\lambda(x) \tau(v)w, \quad (2.23)$$

where $g_{v,w}^\lambda(x)$ are some meromorphic functions and $\tau(v)$ are the shift operators acting on functions by $\tau(v)f(x) = f(x - \hbar v)$.

The subspace $\mathcal{F}^W \subset \mathcal{F}$ can be characterized as a space annihilated by all the operators $T_1 - t_1, \dots, T_n - t_n$. Since the elements of $\mathbb{C}[Y]^W$ commute with T_1, \dots, T_n they preserve this subspace: $m_\lambda(Y)(\mathcal{F}^W) \subset \mathcal{F}^W$. Thus we can restrict the elements of $\mathbb{C}[Y]^W$ to \mathcal{F}^W [5,6]. The restriction of (2.23) is

$$M_\lambda = \text{Res}(m_\lambda(Y)) = m_\lambda(Y)|_{\mathcal{F}^W} = \sum_{v \in P^\vee} \sum_{w \in W} g_{v,w}^\lambda(x) \tau(v). \quad (2.24)$$

Let $\text{Mer}(V)$ be the space of meromorphic functions $\varphi(x)$ on V such that $\varphi(x + 2\pi i \alpha_j^\vee) = \varphi(x)$ for all $j = 1, \dots, n$. Define the subspace $\mathcal{M} \subset \text{Mer}(V)$ consisting of functions $\varphi \in \text{Mer}(V)$ holomorphic out of the union of hyperplanes $\bigcup_{\beta \in \widehat{R}} \{x \in V \mid \beta(x) \in 2\pi i \mathbb{Z}\}$. Consider the smash-product algebra $\mathbb{D} = \mathcal{M} \# \mathbb{C}[\tau(P^\vee)]$ defined by the commutation relation $\tau(\lambda)\varphi(x) = \varphi(x - \hbar\lambda)\tau(\lambda)$. This is the algebra of difference operators acting on the space \mathcal{M} . Any element $a \in \mathbb{D}$ has the form $a = \sum_{\lambda \in P^\vee} \sum_{w \in W} g_{\lambda,w}(x) \tau(\lambda)w$, where $g_{\lambda,w} \in \mathcal{M}$ and the sum over λ is finite. Note that $M_\lambda \in \mathbb{D}$ for any $\lambda \in P^\vee$.

Let $r \in \mathcal{O}'$, so $b_r \in P^\vee$ is a minuscule coweight. Then the restriction of $m_{-b_r}(Y)$ to \mathcal{F}^W can be written explicitly [5,8,25]:

$$M_{-b_r} = \sum_{\lambda \in W(-b_r)} \left(\prod_{\substack{\alpha \in R \\ (\alpha, \lambda)=1}} \frac{t_\alpha e^{(\alpha, x)} - t_\alpha^{-1}}{e^{(\alpha, x)} - 1} \right) \tau(\lambda) \quad (2.25)$$

$$= \sum_{\lambda \in W(-b_r)} \left(\prod_{\substack{\alpha \in R \\ (\alpha, \lambda)=1}} \frac{t_\alpha e^{(\alpha, x)} - t_\alpha^{-1}}{e^{(\alpha, x)} - 1} \right) (\tau(\lambda) - 1) + m_{-b_r, t}, \quad (2.26)$$

where $m_{\lambda, t} = \sum_{v \in W_\lambda} \prod_{\alpha \in R_+} t_\alpha^{(\alpha, v)}$. Note that in the cases $R = E_8, F_4, G_2$ the minuscule coweights do not exist. For any R there exists quasi-minuscule coweight $\lambda = -\theta^\vee$. The corresponding operator has the form [8,25]:

$$M_{-\theta^\vee} = \sum_{\beta \in W\theta} \left(\prod_{\substack{\alpha \in R \\ (\alpha, \beta^\vee) > 0}} \frac{t_\alpha e^{(\alpha, x)} - t_\alpha^{-1}}{e^{(\alpha, x)} - 1} \right) \frac{q^{-1} t_0 e^{(\beta, x)} - q t_0^{-1}}{q^{-1} e^{(\beta, x)} - q} (\tau(\beta^\vee) - 1) + m_{-\theta^\vee, t}. \quad (2.27)$$

The operators (2.25), (2.27) are called *Macdonald operators* (see e.g. [24]).

Consider the subspace $\mathfrak{R} \subset \mathbb{D}$ spanned by the elements M_λ defined by (2.24). Since the restriction operation $\text{Res} : m_\lambda(Y) \mapsto M_\lambda$ is an algebra homomorphism the subspace $\mathfrak{R} = \text{Res}(\mathbb{C}[Y]^W)$ is a commutative subalgebra in \mathbb{D} . We will recall that \mathfrak{R} contains n algebraically independent elements. First we introduce some notations. For each $\lambda \in P^\vee$ there exists a unique dominant coweight $\lambda_+ \in P_+^\vee$ such that $\lambda_+ \in W\lambda$. Let $\nu, \lambda \in P^\vee$ be two coweights. We write

$\nu > \lambda$ if $\nu \neq \lambda$ and $\nu - \lambda = \sum_{j=1}^n k_j \alpha_j^\vee$ for some $k_j \in \mathbb{Z}_{\geq 0}$. We also introduce another order on P^\vee : we write $\nu < \lambda$ if $\nu_+ < \lambda_+$ or $(\nu_+ = \lambda_+ \text{ and } \nu > \lambda)$ (see [20]).

As it is explained in [20] the Cherednik–Dunkl operators have the form

$$Y^\lambda = \sum_{w \in W} \tilde{g}_w^\lambda(x) \tau(\lambda) w + \sum_{\nu < \lambda} \sum_{w \in W} \tilde{g}_{\nu, w}^\lambda(x) \tau(\nu) w, \quad (2.28)$$

for some functions $\tilde{g}_{\nu, w}^\lambda(x)$, $\tilde{g}_w^\lambda(x)$. For an anti-dominant coweight $\lambda \in P_-^\vee$ one has a more exact formula [20]

$$Y^\lambda = g^\lambda(x) \tau(\lambda) + \sum_{\nu < \lambda} \sum_{w \in W} \tilde{g}_{\nu, w}^\lambda(x) \tau(\nu) w. \quad (2.29)$$

Here $g^\lambda(x) = \prod_{\alpha \in \hat{R}_\lambda} \frac{t_\alpha e^{\alpha(x)} - t_\alpha^{-1}}{e^{\alpha(x)} - 1}$, where $\hat{R}_\lambda = \hat{R}_- \cap (\tau(\lambda) \hat{R}_+)$ is a subset in \hat{R} . Thus, for any $\lambda \in P_-^\vee$ we obtain

$$m_\lambda(Y) = g^\lambda(x) \tau(\lambda) + \sum_{\nu < \lambda} \sum_{w \in W} g_{\nu, w}^\lambda(x) \tau(\nu) w \quad (2.30)$$

for some functions $\tilde{g}_{\nu, w}^\lambda(x)$. In accordance with the formula (2.24) the restriction of (2.30) to the subspace \mathcal{F}^W gives the following difference operator

$$M_\lambda = g^\lambda(x) \tau(\lambda) + \sum_{\nu < \lambda} g_\nu^\lambda(x) \tau(\nu), \quad (2.31)$$

where $g_\nu^\lambda(x) = \sum_{w \in W} \tilde{g}_{\nu, w}^\lambda(x)$. Since in the case $\lambda \in P_-^\vee$ the condition $\nu < \lambda$ implies $\nu > \lambda$ we can replace the order $<$ with the order $>$ in the sum in the formula (2.31). It follows now that for any $m_1, \dots, m_n \in \mathbb{Z}_{\geq 0}$ one has

$$\prod_{j=1}^n (M_{-b_j})^{m_j} = g^{m_1, \dots, m_n}(x) \tau(\lambda) + \sum_{\nu > \lambda} g_\nu^{m_1, \dots, m_n}(x) \tau(\nu), \quad (2.32)$$

where $g^{m_1, \dots, m_n}(x) \neq 0$ and $\lambda = -\sum_{i=1}^n m_i b_i \in P_-^\vee$. Since the right-hand sides of (2.32) are linearly independent the operators $M_{-b_1}, \dots, M_{-b_n}$ are algebraically independent.

Remark 2.1. One can show that the operator (2.32) has the form $c_\lambda^\lambda M_\lambda + \sum_{\nu \in P_-^\vee, \nu > \lambda} c_\nu^\lambda M_\nu$ such that $c_\lambda^\lambda \neq 0$. Then [1, VI, §3, 4, Lemma 4] implies that the operators (2.32) form a basis of the space \mathfrak{R} . It follows the commutative algebra \mathfrak{R} is isomorphic to the algebra of polynomials $\mathbb{C}[z_1, \dots, z_n]$, and the isomorphism can be given by $M_{-b_j} \mapsto z_j$. In particular the elements $M_{-b_1}, \dots, M_{-b_n}$ generate the algebra \mathfrak{R} .

The subalgebra $\mathfrak{R} \subset \mathbb{D}$ is called the *Macdonald–Ruijsenaars system*. The Macdonald–Ruijsenaars operator (2.25) or the operator (2.27) can be considered as a Hamiltonian of this system.

3. Invariant shifted parabolic ideals

The goal of this section is to construct some \mathcal{H}_t -submodules of \mathcal{F} , which will be used in the next section. Note that any ideal of the algebra \mathcal{F} is invariant under the action of the generators

$X^\mu \in \mathcal{H}_l$. We define some ideals as the sets of functions vanishing on certain planes. We show that these ideals are invariant under the actions of other generators of DAHA if parameters satisfy special conditions.

Let $\eta : R \rightarrow \mathbb{C}$ be a W -invariant function on the irreducible reduced root system R , that is $\eta_{w(\alpha)} = \eta_\alpha = \eta(\alpha)$, $\forall \alpha \in R$, $\forall w \in W$. We will also denote $\eta_i = \eta_{\alpha_i}$ for the simple roots α_i . To each subset $S \subset R$ we associate the plane

$$H_\eta(S) = \{x \in V \mid (\alpha, x) = \eta_\alpha, \forall \alpha \in S\} = \bigcap_{\alpha \in S} \{x \in V \mid (\alpha, x) = \eta_\alpha\}. \quad (3.1)$$

Note that $H_\eta(wS) = wH_\eta(S)$ for any $w \in W$.

Let S_0 be a subset of the set of simple roots Δ and let $W_0 \subset W$ be the subgroup generated by s_α , $\alpha \in S_0$. We associate to S_0 the following subset

$$W_+ = \{w \in W \mid wS_0 \subset R_+\} \subset W. \quad (3.2)$$

Let $V_0 = \langle S_0 \rangle$ be the linear subspace of V spanned by the set S_0 . Then $R_0 = V_0 \cap R$ is the root system with the set of simple roots S_0 . Note that the transformations $w \in W_+$ restricted to R_0 respect the standard order. More exactly, if $\beta \leq \gamma$, $\beta, \gamma \in R_0$ then $w\beta \leq w\gamma$. (Here $\delta_1 \leq \delta_2$ means $\delta_2 - \delta_1 = \sum_{i=1}^n m_i \alpha_i$, where $m_i \in \mathbb{Z}_{\geq 0}$.)

Remark 3.1. The elements of W_+ are the shortest coset representatives of the left cosets of W_0 in W (see [17]).

Let us consider the plane

$$\mathcal{D}_0 = H_\eta(S_0) = \{x \in V \mid (\alpha, x) = \eta_\alpha, \forall \alpha \in S_0\} \quad (3.3)$$

and the union of planes

$$\mathcal{D} = \bigcup_{w \in W_+} H_\eta(wS_0) = W_+ \mathcal{D}_0. \quad (3.4)$$

The arrangement \mathcal{D} is obtained by shifting the linear arrangement $WH_0(S_0)$, where $H_0(S_0)$ is defined by the formula (3.3) with the function $\eta \equiv 0$. More exactly, the following two statements take place.

Proposition 3.1. For any $w \in W$ there exists $w' \in W_+$ such that $H_0(wS_0) = H_0(w'S_0)$. Thus each plane $wH_0(S_0)$ is parallel to some plane $w'\mathcal{D}_0 \subset \mathcal{D}$.

Proof. The element $w \in W$ can be decomposed as $w = w'w_0$ where $w' \in W_+$, $w_0 \in W_0$ (see [17]). Since $H_0(S_0) = H_0(R_0)$ and $w_0R_0 = R_0$ we have $w_0H_0(S_0) = H_0(S_0)$. Hence $H_0(wS_0) = w'H_0(S_0) = w'\mathcal{D}_0$. \square

Proposition 3.2. If the planes $w\mathcal{D}_0$ and $w'\mathcal{D}_0$ are parallel for some $w, w' \in W_+$ then they coincide: $w\mathcal{D}_0 = w'\mathcal{D}_0$.

Proof. Let $w, w' \in W_+$. If the planes $w\mathcal{D}_0$ and $w'\mathcal{D}_0$ are parallel then $wV_0 = w'V_0$. In particular, this implies

$$w\alpha = \sum_{\beta \in S_0} \gamma_{\alpha\beta} w'\beta, \quad \text{for all } \alpha \in S_0, \quad (3.5)$$

where $\gamma_{\alpha\beta} \in \mathbb{C}$. Applying $(w')^{-1}$ to (3.5) we conclude that $\gamma_{\alpha\beta} \in \mathbb{Z}_{\geq 0}$, $\forall \beta \in S_0$ or $\gamma_{\alpha\beta} \in \mathbb{Z}_{\leq 0}$, $\forall \beta \in S_0$. Taking into account $w\alpha \in R_+$ and $w'\beta \in R_+$ we deduce from (3.5) that $\gamma_{\alpha\beta} \in \mathbb{Z}_{\geq 0}$, $\forall \alpha, \beta \in S_0$. Analogously we have

$$w'\beta = \sum_{\alpha \in S_0} \tilde{\gamma}_{\beta\alpha} w\alpha, \quad \text{for all } \beta \in S_0, \quad (3.6)$$

where $\tilde{\gamma}_{\beta\alpha} \in \mathbb{Z}_{\geq 0}$. The formulae (3.5), (3.6) lead to the following system of equations

$$\sum_{\beta \in S_0} \gamma_{\alpha\beta} \tilde{\gamma}_{\beta\tilde{\alpha}} = \delta_{\alpha, \tilde{\alpha}}, \quad \alpha, \tilde{\alpha} \in S_0, \quad (3.7)$$

where $\delta_{\alpha, \beta}$ is the delta-symbol: $\delta_{\alpha, \alpha} = 1$, $\delta_{\alpha, \beta} = 0$ for $\alpha \neq \beta$. Under conditions $\gamma_{\alpha\beta}, \tilde{\gamma}_{\beta\alpha} \in \mathbb{Z}_{\geq 0}$ the general solution of the system (3.7) is

$$\gamma_{\alpha\beta} = \tilde{\gamma}_{\beta\alpha} = \delta_{\sigma(\alpha), \beta}, \quad (3.8)$$

where $\sigma : S_0 \rightarrow S_0$ is a bijection. Hence

$$w\alpha = \sum_{\beta \in S_0} \delta_{\sigma(\alpha), \beta} w'\beta = w'\sigma(\alpha), \quad \text{for all } \alpha \in S_0. \quad (3.9)$$

Using this one yields

$$\begin{aligned} w\mathcal{D}_0 &= \{x \in V \mid (w\alpha, x) = \eta_\alpha, \forall \alpha \in S_0\} = \{x \in V \mid (w'\sigma(\alpha), x) = \eta_\alpha, \forall \alpha \in S_0\} \\ &= \{x \in V \mid (w'\sigma(\alpha), x) = \eta_{\sigma(\alpha)}, \forall \alpha \in S_0\} = \{x \in V \mid (w'\alpha, x) = \eta_\alpha, \forall \alpha \in S_0\} \\ &= w'\mathcal{D}_0, \end{aligned}$$

where we used $\eta_\alpha = \eta_{\sigma(\alpha)}$ obtained from (3.9). \square

Let R_1, \dots, R_m be irreducible components of $R_0 = V_0 \cap R$, $R_0 = \bigsqcup_{\ell=1}^m R_\ell$. In each root subsystem R_ℓ we have the corresponding set of simple roots $S_\ell = S_0 \cap R_\ell$, so that $S_0 = \bigsqcup_{\ell=1}^m S_\ell$, where $S_\ell \cap S_{\ell'} = \emptyset$ for $\ell \neq \ell'$. The reflections s_α , $\alpha \in R_\ell$, generate the Weyl group $W_\ell \subset W$ corresponding to the root system R_ℓ . For each root system R_ℓ there exists the maximal root $\theta_\ell \in R_\ell$ with respect to S_ℓ . Let $k_\ell = |S_\ell|$ and $S_\ell = \{\alpha_1^\ell, \alpha_2^\ell, \dots, \alpha_{k_\ell}^\ell\}$, then $\theta_\ell = \sum_{j=1}^{k_\ell} n_j^\ell \alpha_j^\ell$, where $n_j^\ell \geq 1$ are integers.

Definition 3.1. Let $\theta = \sum_{i=1}^n n_i \alpha_i$ be the maximal root of R with respect to the set of simple roots Δ . The number

$$h_\eta = \eta_\theta + \sum_{i=1}^n n_i \eta_{\alpha_i}$$

is called generalized Coxeter number for the root system R and the invariant function η .

We note that the generalized Coxeter number for the root system R does not depend on the choice of R_+ (or, equivalently, on the choice of Δ). For a constant function $\eta_\alpha \equiv \eta$ the generalized Coxeter number $h_\eta = \eta \cdot h$, where $h = 1 + \sum_{i=1}^n n_i$ is the usual Coxeter number (see [1]).

Remark 3.2. The generalized Coxeter number can be equivalently defined as the coefficient of proportionality of the following W -invariant inner products (cf. [13]):

$$\sum_{\alpha \in R} \frac{\eta_{\alpha}(\alpha, u)(\alpha, v)}{(\alpha, \alpha)} = h_{\eta} \cdot (u, v), \quad \forall u, v \in V. \quad (3.10)$$

Note that the restriction of the function $\eta : R \rightarrow \mathbb{C}$ to the subsystems $R_{\ell} \subset R$ is a W_{ℓ} -invariant function on R_{ℓ} . Since R_{ℓ} is irreducible one can define the generalized Coxeter numbers for this subsystem:

$$h_{\eta}^{\ell} = \eta_{\theta_{\ell}} + \sum_{j=1}^{k_{\ell}} n_j^{\ell} \eta_j^{\ell}, \quad (3.11)$$

where $\eta_j^{\ell} = \eta_{\alpha_j^{\ell}}$.

Define the ideal of functions vanishing on the union of planes (3.4):

$$I_{\mathcal{D}} = \{f \in \mathcal{F} \mid f(x) = 0, \forall x \in \mathcal{D}\}. \quad (3.12)$$

The main result of this section is the following theorem.

Theorem 3.3. Suppose that the subsystem $S_0 \subset \Delta$ and the function η are such that

$$h_{\eta}^{\ell} = \hbar, \quad \text{for all } \ell = 1, \dots, m. \quad (3.13)$$

Let the parameters of DAHA \mathcal{H}_t satisfy $t_{\alpha}^2 = e^{\eta_{\alpha}}$ for all $\alpha \in R$. Then the ideal $I_{\mathcal{D}} \subset \mathcal{F}$ is invariant under the action of DAHA: $\mathcal{H}_t I_{\mathcal{D}} \subset I_{\mathcal{D}}$.

We will prove this theorem in the next three subsections. We establish the invariance under T_i , $1 \leq i \leq n$ in Section 3.1, the invariance under π_r is established in Section 3.2 and the invariance under T_0 is established in Section 3.3.

Remark 3.3. The degenerations of the ideals $I_{\mathcal{D}}$ at $\eta \rightarrow 0$ give invariant parabolic ideals for the rational Cherednik algebra $H_c(W)$ with $c = \eta/\hbar$. Any radical ideal invariant under $H_c(W)$ can be obtained this way (see [13]).

Remark 3.4. The invariant ideals $I_{\mathcal{D}}$ for the A series are the ideals obtained by Kasatani in [18], namely these are the ideals $I_m^{(k,r)}$ with $r = 2$ and the same m as we use. (The ideals in [18] are considered in the space of polynomials of multiplicative variables.)

Let us record the introduced planes in terms of affine functionals. For a root $\alpha \in R_+$ we denote $\alpha' = [\alpha, -\eta_{\alpha}] \in \widehat{V}$. Note that $w(\alpha') = [w\alpha, -\eta_{\alpha}] = [w\alpha, -\eta_{w\alpha}] = (w\alpha)'$ for all $w \in W_+$, $\alpha \in R_+$. For a subset $S \subset \widehat{V}$ let

$$H(S) = \{x \in V \mid \beta(x) = 0, \forall \beta \in S\}. \quad (3.14)$$

We will write $H(\beta_1, \beta_2, \dots)$ instead of $H(\{\beta_1, \beta_2, \dots\})$. In these notations the definitions (3.1) and (3.4) take the form

$$H_{\eta}(S) = H(S') = \{x \in V \mid \alpha'(x) = 0, \forall \alpha \in S\}, \quad \mathcal{D} = \bigcup_{w \in W_+} H(wS'_0), \quad (3.15)$$

where $S' = \{\alpha' \mid \alpha \in S\}$, $S'_0 = \{\alpha' \mid \alpha \in S_0\}$. Note also that $wH(S') = H(wS')$.

3.1. Invariance under T_1, \dots, T_n

Lemma 3.4. Let $S = wS_0$ where $S_0 \subset \Delta$, and $w \in W_+$. Let $\beta \in \Delta$. If $\beta \notin S$ then β cannot be expressed as a linear combination of the elements of S .

Proof. Suppose that there exist numbers $\gamma_\alpha \in \mathbb{C}$ for $\alpha \in S_0$ such that $\beta = \sum_{\alpha \in S_0} \gamma_\alpha w\alpha$. Then, $w^{-1}(\beta) = \sum_{\alpha \in S_0} \gamma_\alpha \alpha$. Since $S_0 \subset \Delta$ this equation implies that the numbers γ_α are integers with the same sign: $\gamma'_\alpha \geq 0, \forall \alpha \in S_0$ or $\gamma'_\alpha \leq 0, \forall \alpha \in S_0$. Taking into account that $w\alpha \in R_+$ for any $\alpha \in S_0$ we see that the relation $\beta = \sum_{\alpha \in S_0} \gamma_\alpha w\alpha$ holds only if $\beta \in S$, but this contradicts the condition of the lemma. \square

Proposition 3.5. Let parameters of DAHA \mathcal{H}_t satisfy $t_\alpha^2 = e^{\eta_\alpha}$ for all $\alpha \in R$. Then the ideal $I_{\mathcal{D}}$ given by (3.12), (3.4), (3.3) is invariant under the action of the operators T_1, \dots, T_n , moreover $T_i I_{\mathcal{D}} = I_{\mathcal{D}}$ for all $i = 1, \dots, n$.

Proof. Let $f \in I_{\mathcal{D}}$. Consider the subset $S = wS_0 \subset R_+$, where $w \in W_+$. We need to prove that the function $T_i f(x)$ vanishes on the plane $H(S')$.

Consider first the case $\alpha_i \in S$. Consider the expression (2.21) for the function $T_i f(x)$. Let $x \in H(S')$, then the first term in this expression vanishes since the function $f(x)$ vanishes on the plane $H(S')$ and the second term vanishes since $t_i^{-1} - t_i e^{-\alpha_i(x)} = t_i^{-1} - t_i e^{-\eta_i} = 0$.

Now consider the case $\alpha_i \notin S$. Then we have $s_i(S) \subset R_+$. This means that $s_i w \in W_+$, $s_i H(S') = H(s_i w S_0) \subset \mathcal{D}$ and $f(s_i x) = 0$ whenever $x \in H(S')$. Thus, if $x \in H(S')$ and $e^{-\alpha_i(x)} \neq 1$ then both terms in

$$T_i f(x) = \frac{t_i - t_i^{-1}}{1 - e^{-\alpha_i(x)}} f(x) + \frac{t_i^{-1} - t_i e^{-\alpha_i(x)}}{1 - e^{-\alpha_i(x)}} f(s_i x) \quad (3.16)$$

vanish: $T_i f(x) = 0$. Note that for any $p \in \mathbb{Z}$ the intersection of the planes $H(S')$ and $H([\alpha_i, 2\pi i p])$ has positive codimension in $H(S')$. Indeed, otherwise $H(S') \subset H([\alpha_i, 2\pi i p])$ and this implies, in particular, that α_i can be obtained as a linear combination of the elements of S , but this is impossible by Lemma 3.4. Thus the subset $\{x \in H(S') \mid e^{-\alpha_i(x)} \neq 1\}$ is everywhere dense in the plane $H(S')$. Since the function $T_i f(x)$ is continuous it vanishes on the whole plane: $T_i f(x) = 0, \forall x \in H(S')$.

We have proved $T_i I_{\mathcal{D}} \subset I_{\mathcal{D}}$. The relation $T_i^{-1} = T_i + (t_i^{-1} - t_i)$ implies $T_i^{-1} I_{\mathcal{D}} \subset I_{\mathcal{D}}$ and then $T_i I_{\mathcal{D}} = I_{\mathcal{D}}$. \square

Remark 3.5. If the W -invariant function $\eta : R \rightarrow \mathbb{C}$ is fixed then one can choose a \widehat{W} -invariant function $t : \widehat{R} \rightarrow \mathbb{C}$ satisfying the conditions of Proposition 3.5 by the formula $t_{[\beta, hk]} = e^{\eta_\beta/2}$ for all $\beta \in R$ and $k \in \mathbb{Z}$.

3.2. Invariance under the group Π

Lemma 3.6. (Cf. [1].) Let $\theta = \sum_{i=1}^n n_i \alpha_i$ be the maximal root of R with respect to the set of simple roots Δ . Then $n_r = 1$ for some r if and only if there exists $u_r \in W$ such that $u_r(\{-\theta\} \cup \Delta) = \{-\theta\} \cup \Delta$ and $u_r(\alpha_r) = -\theta$. Furthermore, if $u_r(\alpha_i) = \alpha_j$ then $n_i = n_j$.

Proof. In the notations of Section 2.1 the statement means $\mathcal{O}' = \{r \mid n_r = 1\}$ and it is proved in [1]. Now let $r \in \mathcal{O}'$. Define the number $r^* \in \mathcal{O}'$ such that $\pi_{r^*} = \pi_r^{-1}$, then $u_{r^*} = u_r^{-1}$,

which leads to $u_r(-\theta) = \alpha_{r^*}$. Taking into account that $u_r(\alpha_r) = -\theta$, $u_r(\alpha_i) = \alpha_j$, $n_r = 1$ and $u_r(\Delta \setminus \{\alpha_r\}) = \Delta \setminus \{\alpha_{r^*}\}$, we rearrange $u_r(-\sum_{i=1}^n n_i \alpha_i) = \alpha_{r^*}$ as

$$n_i \alpha_j + \sum_{k \neq j, r^*}^n \tilde{n}_k \alpha_k - \sum_{k=1}^n n_k \alpha_k + \alpha_{r^*} = 0,$$

where $\tilde{n}_k \in \mathbb{Z}$. We obtain $n_i = n_j$ since $j \neq r^*$ by extracting the coefficient at α_j . \square

Proposition 3.7. Suppose that the subsystem $S_0 \subset \Delta$ and the function η are such that $h_\eta^\ell = \hbar$ for all $\ell = 1, \dots, m$. Then the arrangement (3.4) and the ideal (3.12) satisfy $\pi_r \mathcal{D} = \mathcal{D}$ and $\pi_r I_{\mathcal{D}} = I_{\mathcal{D}}$ for any $r \in \mathcal{O}$.

Proof. It is sufficient to establish that $\pi_r^{-1} \mathcal{D} \subset \mathcal{D}$ for any $r \in \mathcal{O}'$. Consider the subset $wS_0 \subset R_+$, where $w \in W_+$. We will show that there exists $\tilde{w} \in W_+$ such that $\pi_r^{-1} H(wS'_0) = H(\tilde{w}S'_0)$.

First we note that

$$\text{if } \alpha \in R_+ \text{ and } (\alpha, b_r) = 0 \text{ then } u_r \alpha \in R_+. \quad (3.17)$$

Indeed since the transformation π_r^{-1} preserves the set $\tilde{\Delta}$ it also preserves the set \hat{R}_+ . Therefore $u_r \alpha = \pi_r^{-1} \alpha \in \hat{R}_+ \cap R = R_+$, where we used $u_r = \pi_r^{-1} \tau(b_r)$ and $u_r \alpha \in R$.

Lemma 3.6 and $r \in \mathcal{O}'$ imply $n_r = 1$. Since the roots $w\alpha_j^\ell$ and $w\theta_\ell = \sum_{j=1}^{k_\ell} n_j^\ell w\alpha_j^\ell$ are positive and coweight b_r is minuscule the products $(w\theta_\ell, b_r)$ and $(w\alpha_j^\ell, b_r)$ are equal to 0 or 1. So for each ℓ one of the following two possibilities holds:

- (i) $(w\theta_\ell, b_r) = 1$, hence $(w\alpha_j^\ell, b_r) = \delta_{jp_\ell}$ and $n_{p_\ell}^\ell = 1$ for some p_ℓ such that $1 \leq p_\ell \leq k_\ell$;
- (ii) $(w\theta_\ell, b_r) = 0$, hence $(w\alpha_j^\ell, b_r) = 0$ for all $j = 1, \dots, k_\ell$.

We can fix $p_\ell = 1$ by renumbering the roots $\alpha_1^\ell, \dots, \alpha_{k_\ell}^\ell$ if necessary.

Consider first the case (i): $\gamma_{rj}^\ell = \delta_{j1}$ and $n_1^\ell = 1$. Let us first rearrange the plane $H(wS'_\ell)$ as follows. We can replace in

$$H(wS'_\ell) = H([w\alpha_1^\ell, -\eta_1^\ell], [w\alpha_2^\ell, -\eta_2^\ell], \dots, [w\alpha_{k_\ell}^\ell, -\eta_{k_\ell}^\ell])$$

the functional $[w\alpha_1^\ell, -\eta_1^\ell]$ by the linear combination $\sum_{j=1}^{k_\ell} n_j^\ell [-w\alpha_j^\ell, \eta_j^\ell]$; so taking into account $w\theta_\ell = \sum_{j=1}^{k_\ell} n_j^\ell w\alpha_j^\ell$ and the formula (3.11) we obtain

$$H(wS'_\ell) = H([-w\theta_\ell, h_\eta^\ell - \eta_{\theta_\ell}], [w\alpha_2^\ell, -\eta_2^\ell], \dots, [w\alpha_{k_\ell}^\ell, -\eta_{k_\ell}^\ell]). \quad (3.18)$$

This gives

$$\pi_r^{-1} H(wS'_\ell) = H([-u_r w\theta_\ell, h_\eta^\ell - \eta_{\theta_\ell} - \hbar], [u_r w\alpha_2^\ell, -\eta_2^\ell], \dots, [u_r w\alpha_{k_\ell}^\ell, -\eta_{k_\ell}^\ell]) \quad (3.19)$$

using $\pi_r^{-1} = u_r \tau(-b_r)$ and $u_r \tau(-b_r) \alpha = [u_r \alpha^{(1)}, \alpha^{(0)} + \hbar(\alpha^{(1)}, b_r)]$, where $\alpha = [\alpha^{(1)}, \alpha^{(0)}] \in \hat{V}$. It follows from (3.17) that $u_r w\alpha_j^\ell \in R_+$ for $j = 2, \dots, k_\ell$.

We note that $-u_r w\theta_\ell \in R_+$ since $(u_r w\theta_\ell, u_r b_r) = 1$ and $u_r b_r \in P_-^\vee$. The latter property is proved as follows. Since Π is a group there exists $r^* \in \mathcal{O}'$ such that $\pi_r^{-1} = \pi_{r^*}$. Rearranging $\pi_r = \pi_{r^*}^{-1}$ as $\tau(b_r) u_r^{-1} = u_{r^*} \tau(-b_{r^*})$ we obtain $u_r b_r = -b_{r^*} \in P_-^\vee$, where we used the relation $u_r \tau(b_r) u_r^{-1} = \tau(u_r b_r)$.

Since $n_1^\ell = 1$ it follows by Lemma 3.6 applied to the root system R_ℓ that there exists an element $v_\ell \in W_\ell$ such that $v_\ell(S_\ell) = \{-\theta_\ell, \alpha_2^\ell, \dots, \alpha_{k_\ell}^\ell\}$. So using $h_\eta^\ell = \hbar$ we obtain

$$\pi_r^{-1} H(wS'_\ell) = H(u_r w v_\ell S'_\ell) \quad \text{and} \quad u_r w v_\ell S_\ell \subset R_+. \quad (3.20)$$

In the case (ii) the formulae (3.20) are valid for $v_\ell = 1$. Indeed $\pi_r^{-1} H(wS'_\ell) = H(u_r w S'_\ell)$ and it follows from (3.17) that $u_r w \alpha_j^\ell \in R_+$ for all $j = 1, \dots, k_\ell$.

Define now $\tilde{w} = u_r w \prod_{\ell=1}^m v_\ell$. Since $v_\ell \in W_\ell$ acts identically on $S_{\ell'}$ for $\ell' \neq \ell$ we derive

$$\tilde{w} S_0 = \bigcup_{\ell=1}^m u_r w v_\ell S_\ell \subset R_+, \quad (3.21)$$

hence $\tilde{w} \in W_+$, and

$$\begin{aligned} \pi_r^{-1} H(wS'_0) &= \bigcap_{\ell=1}^m \pi_r^{-1} H(wS'_\ell) = \bigcap_{\ell=1}^m H(u_r w v_\ell S'_\ell) \\ &= H\left(\bigcup_{\ell=1}^m u_r w v_\ell S'_\ell\right) = H(\tilde{w} S'_0). \end{aligned} \quad (3.22)$$

Thus the invariance of \mathcal{D} and, as consequence, the invariance of $I_{\mathcal{D}}$ are proven. \square

3.3. Invariance under T_0

If $\mathcal{O} \neq \{0\}$ then there exists $r \in \mathcal{O}'$ and the invariance of $I_{\mathcal{D}}$ under T_0 follows from the formula $T_0 = \pi_r^{-1} T_r \pi_r$. Otherwise ($\mathcal{O} = \{0\}$ for $R = E_8, F_4, G_2$) we need to prove the invariance under T_0 separately. We will prove it in the general case. We start with two preliminary lemmas.

Lemma 3.8. *Let $w \in W_+$. Suppose the maximal roots for the system R and the subsystem R_ℓ for some ℓ satisfy $w\theta_\ell \neq \theta$. Then $\theta \notin wR_\ell$.*

Proof. Suppose $\theta \in wR_\ell$ and let $\beta = w^{-1}\theta$. Then $\beta \in R_\ell$ hence $\beta \leq \theta_\ell$. Since $w \in W_+$ and $\beta, \theta_\ell \in R_0$ it follows $\theta = w\beta \leq w\theta_\ell$ (see the comment after the formula (3.2)). Since θ is the maximal root we get $\theta = w\theta_\ell$, which is a contradiction. \square

Lemma 3.9. *Suppose the subsystem $S_0 \subset \Delta$ and the function η are such that $h_\eta^\ell = \hbar$ for all $\ell = 1, \dots, m$. Let $w \in W_+$ and suppose $w\theta_\ell \neq \theta$ for all $\ell = 1, \dots, m$. Then $s_0 H(wS'_0) \subset \mathcal{D}$.*

Proof. We will prove the existence of an element $\tilde{w} \in W_+$ such that $s_0 H(wS'_0) = H(\tilde{w} S'_0)$. First we note that Lemma 3.8 implies $\theta \neq w\alpha_j^\ell$ for all $j = 1, \dots, k_\ell$, $\ell = 1, \dots, m$. Recall that if $\alpha \in R_+$ and $\alpha \neq \theta$ then (α, θ^\vee) equals 0 or 1 (see [1]). Since $w\theta_\ell = \sum_{j=1}^{k_\ell} n_j^\ell w\alpha_j^\ell \neq \theta$ there are only two possibilities for each ℓ :

- (i) $(w\theta_\ell, \theta^\vee) = 1$ and hence $(w\alpha_j^\ell, \theta^\vee) = \delta_{j, p_\ell}$, $n_{p_\ell}^\ell = 1$ for some p_ℓ such that $1 \leq p_\ell \leq k_\ell$;
- (ii) $(w\theta_\ell, \theta^\vee) = 0$ and hence $(w\alpha_j^\ell, \theta^\vee) = 0$ for all $j = 1, \dots, k_\ell$.

Here we took into account that $w\theta_\ell, w\alpha_j^\ell \in R_+$.

Consider the case (i). We can assume $p_\ell = 1$. Rewriting $H(wS'_\ell)$ as (3.18) and acting to this plane by $s_0 = s_\theta \tau(-\theta^\vee)$ one yields

$$s_0 H(wS'_\ell) = H([-s_\theta w\theta_\ell, h_\eta^\ell - \eta_{\theta_\ell}^\ell - \hbar], [s_\theta w\alpha_2^\ell, -\eta_2^\ell], \dots, [s_\theta w\alpha_{k_\ell}^\ell, -\eta_{k_\ell}^\ell]). \quad (3.23)$$

Taking into account that $s_\theta(\alpha) = \alpha - (\alpha, \theta^\vee)\theta$, $\forall \alpha \in R$, we conclude $s_\theta w\alpha_j^\ell = w\alpha_j^\ell \in R_+$ for $j = 2, \dots, k_\ell$ and $-s_\theta w\theta_\ell = \theta - w\theta_\ell \in R_+$. Since $n_1^\ell = 1$ there exists $v_\ell \in W_\ell$ such that $v_\ell(S_\ell) = \{-\theta_\ell, \alpha_2^\ell, \dots, \alpha_{k_\ell}^\ell\}$ (due to Lemma 3.6). Then the condition $h_\eta^\ell = \hbar$ implies

$$s_0 H(wS'_\ell) = H(s_\theta wv_\ell S'_\ell), \quad s_\theta wv_\ell S_\ell \subset R_+. \quad (3.24)$$

In the case (ii) the formulae (3.24) are valid for $v_\ell = 1$, because in this case $s_0 w\alpha_j^\ell = s_\theta w\alpha_j^\ell = w\alpha_j^\ell \in R_+$ for all $j = 1, \dots, k_\ell$.

Thus we have $s_0 H(wS'_\ell) = H(\tilde{w}S'_\ell)$, where $\tilde{w} = s_\theta w \prod_{\ell=1}^m v_\ell \in W_+$. \square

Now the invariance of the ideal $I_{\mathcal{D}}$ under the action of the operator T_0 can be established.

Proposition 3.10. *Suppose the subsystem $S_0 \subset \Delta$ and the functions η , t satisfy $h_\eta^\ell = \hbar$ for all $\ell = 1, \dots, m$ and $t_0^2 = e^{\eta_\theta}$. Then $T_0 I_{\mathcal{D}} = I_{\mathcal{D}}$.*

Proof. We need to prove that for any $f \in I_{\mathcal{D}}$ and $w \in W_+$ the function $T_0 f(x)$ vanishes on the plane $H(S')$, where $S = wS_0$.

Let us first suppose that $w\theta_\ell = \theta$ for some ℓ . Let $x \in H(S')$. Since $h_\eta^\ell = \hbar$ and $\eta_{\theta_\ell} = \eta_\theta$ one yields $\alpha_0(x) = -(w\theta_\ell, x) + \hbar = \eta_{\theta_\ell} - h_\eta^\ell + \hbar = \eta_\theta$ and hence $e^{-\alpha_0(x)} = t_0^{-2}$. Therefore the right-hand side in (2.21) with $i = 0$ vanishes.

Otherwise $w\theta_\ell \neq \theta$ for all ℓ and by virtue of Lemma 3.9 we have $s_0 H(S') \subset \mathcal{D}$. Hence the function

$$T_0 f(x) = \frac{t_0 - t_0^{-1}}{1 - e^{-\alpha_0(x)}} f(x) + \frac{t_0^{-1} - t_0 e^{-\alpha_0(x)}}{1 - e^{-\alpha_0(x)}} f(s_0 x) \quad (3.25)$$

vanishes on the subset $H(S') \cap \{x \in V \mid e^{-\alpha_0(x)} \neq 1\}$. Let us prove that this subset is everywhere dense in $H(S')$. Indeed, otherwise $H(S')$ is contained in a plane $H(\alpha_0 + [0, 2\pi i p])$ for some $p \in \mathbb{Z}$. This leads to $[-\theta, \hbar + 2\pi i p] = \sum_{\alpha \in wS_0} \gamma_\alpha \alpha'$. Therefore $\theta \in wV_0$, where $V_0 \subset V$ is the subspace spanned by S_0 . Since $w^{-1}\theta \in R$ then $w^{-1}\theta \in R_0 = V_0 \cap R$, and the formula $R_0 = \bigsqcup_{\ell=1}^m R_\ell$ implies that $w^{-1}\theta \in R_\ell$ for some ℓ . But this is impossible due to Lemma 3.8. Thus the continuous function $T_0 f(x)$ vanishes on all the plane $H(S')$. \square

Theorem 3.3 follows from Propositions 3.5, 3.7, 3.10.

3.4. Further invariant ideals

In this subsection we generalize previously constructed ideals $I_{\mathcal{D}}$ by requiring functions to vanish on the families of parallel planes containing the planes wD_0 with $w \in W_+$. For DAHA associated with the classical root systems such submodules were considered by Kasatani in [18,19].

Consider a subset $wS_0 \subset R$, where $w \in W$, and its decomposition $wS_0 = \bigsqcup_{\ell=1}^m wS_\ell$. For a function $\xi : wS_0 \rightarrow \mathbb{Z}_{\geq 0}$ we denote $|\xi|_\ell^w = \sum_{\alpha \in wS_\ell} \xi(\alpha)$, where $\ell = 1, \dots, m$. Note that if

$wS_0 = w'S_0$ for some $w' \in W$ then the values $|\xi|_1^w, \dots, |\xi|_m^w$ and $|\xi|_1^{w'}, \dots, |\xi|_m^{w'}$ are related by $|\xi|_{\ell'}^{w'} = |\xi|_{\sigma(\ell)}^w$, where a permutation $\sigma \in \mathfrak{S}_m$ is such that $R_{\sigma(\ell)} \cong R_\ell$.

Let $r_1, \dots, r_m \in \mathbb{Z}_{\geq 0}$ be such that $r_\ell = r_{\ell'}$ if $R_\ell \cong R_{\ell'}$. For an element $w \in W$ we define the following set of functions

$$\mathcal{E}_{\vec{r}}(w) = \left\{ \xi : wS_0 \rightarrow \mathbb{Z}_{\geq 0} \mid \xi^{-1}(0) \subset R_+, |\xi|_\ell^w \leq r_\ell, \ell = 1, \dots, m \right\}, \quad (3.26)$$

where $\vec{r} = (r_1, \dots, r_m)$. The condition $\xi^{-1}(0) \subset R_+$ means that $\xi(\alpha) = 0$ implies $\alpha \in R_+$.

Let $\eta : R \rightarrow \mathbb{C}$ be a fixed W -invariant function. To a subset $S \subset R$ and a function $\xi : S \rightarrow \mathbb{Z}_{\geq 0}$ we associate the plane

$$H_\eta(S; \xi) = \{x \in V \mid (\alpha, x) = \eta_\alpha - \hbar \xi(\alpha), \forall \alpha \in S\}. \quad (3.27)$$

Starting from the subset S_0 and \vec{r} we define the union of planes

$$\mathcal{D}_{\vec{r}} = \bigcup_{w \in W} \bigcup_{\xi \in \mathcal{E}_{\vec{r}}(w)} H_\eta(wS_0; \xi). \quad (3.28)$$

We will assume that \vec{r} is such that

$$\text{GCD}(r_\ell + 1, k_\ell + 1) = 1, \quad \text{if } R_\ell \cong A_{k_\ell}, \ell = 1, \dots, m, \quad (3.29)$$

$$r_\ell = 0, \quad \text{if } R_\ell \not\cong A_{k_\ell}, \ell = 1, \dots, m, \quad (3.30)$$

where GCD stands for the greatest common divisor. Note that in the case $\vec{r} = 0$ the union $\mathcal{D}_{\vec{r}}$ coincides with $\mathcal{D} = W_+ \mathcal{D}_0$ defined by the formula (3.4). We also generalize the conditions $h_\eta^\ell = \hbar$ as

$$h_\eta^\ell = (r_\ell + 1)\hbar. \quad (3.31)$$

Theorem 3.11. Suppose the parameters of DAHA \mathcal{H}_t satisfy $t_\alpha^2 = e^{\eta_\alpha}$ for all $\alpha \in R$. Suppose the subsystem $S_0 \subset \Delta$, the function η and the parameters r_1, \dots, r_m are such that the conditions (3.29), (3.30) and (3.31) are satisfied. Then the ideal

$$I_{\mathcal{D}_{\vec{r}}} = \{f \in \mathcal{F} \mid f(x) = 0, \text{ for all } x \in \mathcal{D}_{\vec{r}}\} \quad (3.32)$$

where $\mathcal{D}_{\vec{r}}$ is given by (3.28), is invariant under the action of DAHA \mathcal{H}_t .

First we establish the invariance under T_1, \dots, T_n .

Lemma 3.12. Under assumptions of Theorem 3.11 one has $T_i I_{\mathcal{D}_{\vec{r}}} \subset I_{\mathcal{D}_{\vec{r}}}$ for $i = 1, \dots, n$.

Proof. Similar to the proof of Proposition 3.5 we show that the function $(T_i f)(x)$ vanishes on the plane $H_\eta(wS_0; \xi)$, where $f \in I_{\mathcal{D}_{\vec{r}}}$, $w \in W$ and $\xi \in \mathcal{E}_{\vec{r}}(w)$. If $\alpha_i \in \xi^{-1}(0)$ then it follows vanishing of the function $t_i^{-1} - t_i e^{-\alpha_i(x)}$ (see the formula (2.21)). Suppose now that $\alpha_i \notin \xi^{-1}(0)$. It follows $s_i(\xi^{-1}(0)) = (\xi \circ s_i)^{-1}(0) \subset R_+$ and, therefore, $\xi \circ s_i \in \mathcal{E}_{\vec{r}}(s_i w)$. If $x \in H_\eta(wS_0; \xi)$ then $s_i x \in H_\eta(s_i wS_0; \xi \circ s_i) \subset \mathcal{D}_{\vec{r}}$ and the function $f(s_i x)$ vanishes as well as the function $f(x)$. We are left to prove that $H_\eta(wS_0; \xi) \not\subset H([\alpha_i, 2\pi i p])$ for all $p \in \mathbb{Z}$. Suppose the opposite, that is $\alpha_i = \sum_{\alpha \in S_0} \gamma_\alpha w\alpha$ and $2\pi i p = \sum_{\alpha \in S_0} \gamma_\alpha (\hbar \xi(w\alpha) - \eta_\alpha)$ for some $\gamma_\alpha \in \mathbb{C}$ and $p \in \mathbb{Z}$. Since $w^{-1}\alpha_i = \sum_{\alpha \in S_0} \gamma_\alpha \alpha \in R_0$, γ_α are integers of the same sign and there exists ℓ , $1 \leq \ell \leq m$ such that $w^{-1}\alpha_i \in R_\ell$. Therefore $2\pi i p = \sum_{\alpha \in S_\ell} \gamma_\alpha (\hbar \xi(w\alpha) - \eta_\alpha)$. If $r_\ell = 0$ then $\xi|_{wS_\ell} \equiv 0$ hence

$wS_\ell \subset R_+$ and $\alpha_i \in wS_\ell$ which is not possible as $\xi(\alpha_i) \neq 0$. If $r_\ell \neq 0$ then $R_\ell \cong A_{k_\ell}$, hence $\eta|_{R_\ell} = \text{const}$ and $\theta_\ell = \sum_{\alpha \in S_\ell} \alpha$. Note that $\pm w^{-1}\alpha_i = \pm \sum_{\alpha \in S_\ell} \alpha \leq \theta_\ell$ hence $|\gamma_\alpha| \leq 1$. The condition (3.31) implies

$$2\pi i p = \hbar \left(\sum_{\alpha \in S_\ell} \gamma_\alpha \xi(w\alpha) - \frac{r_\ell + 1}{k_\ell + 1} \sum_{\alpha \in S_\ell} \gamma_\alpha \right). \quad (3.33)$$

Since $\hbar \notin 2\pi i\mathbb{Q}$ the expression in the brackets vanishes. As $|\sum_{\alpha \in S_\ell} \gamma_\alpha| = \sum_{\alpha \in S_\ell} |\gamma_\alpha| \leq k_\ell$ we get a contradiction with the condition (3.29). \square

We will need the following technical lemma for the rest of the proof of Theorem 3.11.

Lemma 3.13. *Let $u \in W$, $\lambda \in P_+^\vee$ such that $u\lambda \in P_-^\vee$. Suppose that $\hat{u}R_+ \subset \hat{R}_+$, where $\hat{u} = u\tau(-\lambda)$. Let $R_\ell \cong A_{k_\ell}$ for some ℓ and let $S_\ell = \{\alpha_1^\ell, \dots, \alpha_{k_\ell}^\ell\}$ be a basis of simple roots. Assume that $w \in W$ and $\xi \in \Xi_{\tilde{r}}(w)$ satisfy the conditions $|\xi|_\ell^w + m_0^\ell \leq r_\ell + 1$ and $\xi_j^\ell + m_j^\ell \geq 0$ for all $j = 1, \dots, k_\ell$, where $m_0^\ell = (w\theta_\ell, \lambda)$, $\xi_j^\ell = \xi(w\alpha_j^\ell)$, $m_j^\ell = (w\alpha_j^\ell, \lambda)$. Denote $\xi_\ell = \xi|_{wS_\ell}$. Then $\hat{u}H_\eta(wS_\ell; \xi_\ell) = H_\eta(uwv_\ell S_\ell; \tilde{\xi}_\ell)$ for some element $v_\ell \in W_\ell$ and some map $\tilde{\xi}_\ell : uwv_\ell S_\ell \rightarrow \mathbb{Z}_{\geq 0}$ such that $\sum_{\alpha \in uwv_\ell S_\ell} \tilde{\xi}_\ell(\alpha) \leq r_\ell$ and $\tilde{\xi}_\ell^{-1}(0) \subset R_+$.*

Proof. Since $R_\ell \cong A_{k_\ell}$ the maximal root in R_ℓ is $\theta_\ell = \sum_{j=1}^{k_\ell} \alpha_j^\ell$ and hence $m_0^\ell = \sum_{j=1}^{k_\ell} m_j^\ell$. Denoting $\zeta_j^\ell = \xi_j^\ell + m_j^\ell$ we have $\zeta_j^\ell \geq 0$. Let us show that

$$\zeta_j^\ell = 0 \quad \text{implies} \quad uw\alpha_j^\ell \in R_+. \quad (3.34)$$

If $\zeta_j^\ell = 0$ there are two cases: $\xi_j^\ell = m_j^\ell = 0$ and $\xi_j^\ell = -m_j^\ell > 0$. Consider the first case. Note that $m_j^\ell = 0$ implies that $uw\alpha_j^\ell = \hat{u}w\alpha_j^\ell$. Since $\xi_j^\ell = 0$ we have $w\alpha_j^\ell \in R_+$ and using $\hat{u}R_+ \subset \hat{R}_+$ we obtain $uw\alpha_j^\ell \in R \cap \hat{R}_+ = R_+$. In the case when $-m_j^\ell > 0$ we have $(uw\alpha_j^\ell, u\lambda) = m_j^\ell < 0$. Since $u\lambda \in P_-^\vee$ we also deduce $uw\alpha_j^\ell \in R_+$.

There are two possible cases:

- (i) $|\xi|_\ell^w + m_0^\ell = r_\ell + 1$ and hence $m_0^\ell \geq 1$;
- (ii) $|\xi|_\ell^w + m_0^\ell \leq r_\ell$.

Consider first the case (i). Since $m_0^\ell \geq 1$ there exists p_ℓ , $1 \leq p_\ell \leq k_\ell$ such that $m_{p_\ell}^\ell \geq 1$. We can assume $p_\ell = 1$, that is $m_1^\ell \geq 1$. The formula (3.18) can be generalized in this case as

$$H_\eta(wS_\ell; \xi_\ell) = H([-w\theta_\ell, -\hbar|\xi|_\ell^w + h_\eta^\ell - \eta_{\theta_\ell}], [w\alpha_2^\ell, \hbar\xi_2^\ell - \eta_2^\ell], \dots, [w\alpha_{k_\ell}^\ell, \hbar\xi_{k_\ell}^\ell - \eta_{k_\ell}^\ell]). \quad (3.35)$$

Acting by $\hat{u} = u\tau(-\lambda)$ and taking into account (3.31) we obtain

$$\hat{u}H_\eta(wS_\ell; \xi_\ell) = H([-uw\theta_\ell, -\eta_{\theta_\ell}], [uw\alpha_2^\ell, \hbar\xi_2^\ell - \eta_2^\ell], \dots, [uw\alpha_{k_\ell}^\ell, \hbar\xi_{k_\ell}^\ell - \eta_{k_\ell}^\ell]). \quad (3.36)$$

Due to Lemma 3.6 there exists $v_\ell \in W_\ell$ such that $\{-\theta_\ell, \alpha_2^\ell, \dots, \alpha_{k_\ell}^\ell\} = v_\ell S_\ell$. Define the function $\tilde{\xi}_\ell : uwv_\ell S_\ell \rightarrow \mathbb{Z}_{\geq 0}$ by $\tilde{\xi}_\ell(-uw\theta_\ell) = 0$ and $\tilde{\xi}_\ell(uw\alpha_j^\ell) = \zeta_j^\ell$ where $j = 2, \dots, k_\ell$. Let us check

that the sum of values of the function $\tilde{\xi}_\ell$ is not greater than r_ℓ . Indeed, $\sum_{j=2}^{k_\ell} \zeta_j^\ell = |\xi|_\ell^w + m_0^\ell - \xi_1^\ell - m_1^\ell = r_\ell + 1 - \xi_1^\ell - m_1^\ell \leq r_\ell$, where we used $m_1^\ell \geq 1$. Since $u\lambda \in P_-^\vee$ and $(uw\theta_\ell, u\lambda) = m_0^\ell \geq 1$ we have $-uw\theta_\ell \in R_+$. Taking into account (3.34) we derive $\tilde{\xi}_\ell^{-1}(0) \subset R_+$.

In the case (ii) we set $v_\ell = 1$ and $\tilde{\xi}_\ell(uw\alpha_j^\ell) = \zeta_j^\ell$ where $j = 1, \dots, k_\ell$. The sum of values of the function $\tilde{\xi}_\ell$ is still not greater than r_ℓ , indeed $\sum_{j=1}^{k_\ell} \zeta_j^\ell = |\xi|_\ell^w + m_0^\ell \leq r_\ell$. From (3.34) we derive $\tilde{\xi}_\ell^{-1}(0) \subset R_+$ for this case as well.

In both cases we arrive at $\hat{u}H_\eta(wS_\ell; \xi_\ell) = H_\eta(uwv_\ell S_\ell; \tilde{\xi}_\ell)$. \square

Now we are ready to establish the invariance of $I_{\mathcal{D}_\tau}$ under the operators π_r and T_0 .

Lemma 3.14. *Let $r \in \mathcal{O}$. Under assumptions of Theorem 3.11 one has $\pi_r I_{\mathcal{D}_\tau} \subset I_{\mathcal{D}_\tau}$.*

Proof. It is sufficient to establish $\pi_r^{-1}\mathcal{D}_\tau \subset \mathcal{D}_\tau$ for arbitrary $r \in \mathcal{O}'$. Consider an element $w \in W$ and a function $\xi \in \mathcal{E}_\tau(w)$. We will show that there exists an element $\tilde{w} \in W$ and a function $\tilde{\xi} \in \mathcal{E}_\tau(\tilde{w})$ such that $\pi_r^{-1}H_\eta(wS_0; \xi) = H_\eta(\tilde{w}S_0; \tilde{\xi})$. Note that we have

$$\pi_r^{-1}H_\eta(wS_0; \xi) = \bigcap_{\ell=1}^m \pi_r^{-1}H_\eta(wS_\ell; \xi_\ell), \quad (3.37)$$

where $\xi_\ell = \xi|_{wS_\ell}$. For each ℓ we have two possible cases: $R_\ell \cong A_{k_\ell}$ or $R_\ell \not\cong A_{k_\ell}$.

Suppose first that $R_\ell \cong A_{k_\ell}$. For $u = u_r$ and $\lambda = b_r$ we can apply Lemma 3.13. Indeed, it is easy to see that $u_r b_r \in P_-^\vee$ (see the proof of Proposition 3.7). The transformation $\hat{u} = u_r \tau(-b_r) = \pi_r^{-1}$ preserves the set $\tilde{\Delta}$ and hence $\hat{u}R_+ \subset \hat{R}_+$. Since b_r is a minuscule coweight $m_i^\ell \in \{-1, 0, 1\}$ for $i = 0, 1, \dots, k_\ell$ (we use the notations of Lemma 3.13) and hence $|\xi|_\ell^w + m_0^\ell \leq r_\ell + 1$. Further we have $\xi_j^\ell + m_j^\ell \geq 0$ because otherwise $\xi_j^\ell = 0$ and $m_j^\ell = -1$; since $\xi^{-1}(0) \subset R_+$ the first equality means that $w\alpha_j^\ell \in R_+$, but this contradicts the second equality.

So due to Lemma 3.13 there exists an element $v_\ell \in W_\ell$ and a map $\tilde{\xi}_\ell : u_r w v_\ell S_\ell \rightarrow \mathbb{Z}_{\geq 0}$ with the sum of values not greater than r_ℓ such that $\tilde{\xi}_\ell^{-1}(0) \subset R_+$ and

$$\pi_r^{-1}H_\eta(wS_\ell; \xi_\ell) = H_\eta(u_r w v_\ell S_\ell; \tilde{\xi}_\ell). \quad (3.38)$$

In the case $R_\ell \not\cong A_{k_\ell}$ we have $r_\ell = 0$ and therefore $\xi_\ell \equiv 0$, $H_\eta(wS_\ell; \xi_\ell) = H(wS'_\ell)$. So the relation (3.20) holds for an appropriate $v_\ell \in W_\ell$ (see the proof of Proposition 3.7). In this case the equality (3.38) is valid for the function $\tilde{\xi}_\ell : u_r w v_\ell S_\ell \rightarrow \mathbb{Z}_{\geq 0}$ defined as $\tilde{\xi}_\ell \equiv 0$. Note that we have $\tilde{\xi}_\ell^{-1}(0) = u_r w v_\ell S_\ell \subset R_+$.

Define the element $\tilde{w} = u_r w \prod_{\ell=1}^m v_\ell$ and the function $\tilde{\xi} : \tilde{w}S_0 \rightarrow \mathbb{Z}_{\geq 0}$ by the formula $\tilde{\xi}|_{\tilde{w}S_\ell} = \tilde{\xi}_\ell$. We already checked in fact that $|\tilde{\xi}|_\ell^w \leq r_\ell$. Since $\tilde{\xi}_\ell^{-1}(0) \subset R_+$ for all $\ell = 1, \dots, m$ one has $\tilde{\xi}^{-1}(0) \subset R_+$. So $\tilde{\xi} \in \mathcal{E}_\tau(\tilde{w})$. Thus we obtain $\pi_r^{-1}H_\eta(wS_0; \xi) = H_\eta(\tilde{w}S_0; \tilde{\xi})$ as required (cf. (3.22)). \square

Lemma 3.15. *Under assumptions of Theorem 3.11 one has $T_0 I_{\mathcal{D}_\tau} \subset I_{\mathcal{D}_\tau}$.*

Proof. Let $f \in I_{\mathcal{D}_\tau}$, $w \in W$ and $\xi \in \mathcal{E}_\tau(w)$. Then we need to prove that the function $(T_0 f)(x)$ vanishes on the plane $H_\eta(wS_0; \xi)$. Let ξ_ℓ , α_j^ℓ and ξ_j^ℓ be defined as in Lemma 3.13. Then there are three different possibilities:

- (a) $\theta = w\theta_\ell$ and $|\xi|_\ell^w = r_\ell$ for some ℓ ;
- (b) $w\alpha_j^\ell = -\theta$ and $\xi_j^\ell = 1$ for some j and ℓ ;
- (c) the other cases satisfying neither the condition (a) nor (b).

In the cases (a) and (b) we have $\alpha_0(x) = \eta_\theta$ for $x \in H_\eta(wS_0; \xi)$. So that the expression (2.21) with $i = 0$ for the function $(T_0 f)(x)$ vanish on the plane $H_\eta(wS_0; \xi)$.

Now consider the case (c). We will prove in this case $s_0 H_\eta(wS_0; \xi) \subset \mathcal{D}_\tau$ by providing $\tilde{w} \in W$ and $\tilde{\xi} \in \mathcal{E}_\tau(\tilde{w})$ such that $s_0 H_\eta(wS_0; \xi) = H_\eta(\tilde{w}S_0; \tilde{\xi})$.

Let $R_\ell \cong A_{k_\ell}$. Put $m_0^\ell = (w\theta_\ell, \theta^\vee)$ and $m_j^\ell = (w\alpha_j^\ell, \theta^\vee)$, then $m_0^\ell = \sum_{j=1}^{k_\ell} m_j^\ell$. We have $|m_i^\ell| \leq 2$ for all $i = 0, 1, \dots, k_\ell$. Firstly note that $|\xi|_\ell^w + m_0^\ell \leq r_\ell + 1$. Indeed, in the case (c) we have $\theta \neq w\theta_\ell$ or $|\xi|_\ell^w \leq r_\ell - 1$. If the first condition holds then the inequality we are proving follows from $m_0^\ell \leq 1$ and $|\xi|_\ell^w \leq r_\ell$, otherwise it follows from $m_0^\ell \leq 2$ and $|\xi|_\ell^w \leq r_\ell - 1$.

Secondly $\xi_j^\ell + m_j^\ell \geq 0$ for all $j = 1, \dots, k_\ell$. Indeed, if $\xi_j^\ell + m_j^\ell < 0$ then $m_j^\ell < 0$, which implies $w\alpha_j^\ell \in R_-$ and hence $\xi_j^\ell \neq 0$; so that the only possibility is $\xi_j^\ell = 1$ and $m_j^\ell = -2$. This implies $w\alpha_j^\ell = -\theta$, which is impossible in the case (c).

Note that $s_0 = s_\theta \tau(-\theta^\vee)$ satisfies $s_0 R_+ \subset \widehat{R}_+$. Thus using Lemma 3.13 for $u = s_\theta$ and $\lambda = \theta^\vee$ we have

$$s_0 H_\eta(wS_\ell; \xi_\ell) = H_\eta(s_\theta wv_\ell S_\ell; \tilde{\xi}_\ell) \quad (3.39)$$

for some element $v_\ell \in W_\ell$ and some map $\tilde{\xi}_\ell : s_\theta wv_\ell S_\ell \rightarrow \mathbb{Z}_{\geq 0}$ with sum of values not greater than r_ℓ such that $\tilde{\xi}_\ell^{-1}(0) \subset R_+$.

If $R_\ell \not\cong A_{k_\ell}$ then $r_\ell = 0$ and, therefore, $\xi_\ell \equiv 0$, $H_\eta(wS_\ell; \xi_\ell) = H(wS'_\ell)$. So we have the equality (3.24) for the appropriate $v_\ell \in W_\ell$ (see the proof of Lemma 3.9). Thus in this case the relation (3.39) is valid for the map $\tilde{\xi}_\ell : s_\theta wv_\ell S_\ell \rightarrow \mathbb{Z}_{\geq 0}$ defined as $\tilde{\xi}_\ell \equiv 0$. Note that we have $\tilde{\xi}_\ell^{-1}(0) = s_\theta wv_\ell S_\ell \subset R_+$.

Define the element $\tilde{w} = s_\theta w \prod_{\ell=1}^m v_\ell$ and the function $\tilde{\xi} : \tilde{w}S_0 \rightarrow \mathbb{Z}_{\geq 0}$ by the formula $\tilde{\xi}|_{\tilde{w}S_\ell} = \tilde{\xi}_\ell$. We already checked that $|\tilde{\xi}|_\ell^w \leq r_\ell$. Since $\tilde{\xi}_\ell^{-1}(0) \subset R_+$ for all $\ell = 1, \dots, m$ one has $\tilde{\xi}(0) \in R_+$, so $\tilde{\xi} \in \mathcal{E}_\tau(\tilde{w})$. Then we obtain $s_0 H_\eta(wS_0; \xi) = H_\eta(\tilde{w}S_0; \tilde{\xi})$.

Thus $f(x) = f(s_0 x) = 0$ for all $x \in H_\eta(wS_0; \xi)$ and we are left to prove that the denominator $1 - e^{-\alpha_0(x)}$ in the formula (3.25) does not vanish identically on the plane $H_\eta(wS_0; \xi)$. Indeed otherwise $\alpha_0 + [0, 2\pi i p] = \sum_{\alpha \in S_0} \gamma_\alpha [w\alpha, \hbar \xi(w\alpha) - \eta_\alpha]$ for some $p \in \mathbb{Z}$ and $\gamma_\alpha \in \mathbb{Z}$. Then there exists ℓ , $1 \leq \ell \leq m$ such that $\gamma_\alpha \neq 0$ only for $\alpha \in S_\ell$. Hence $2\pi i p = \hbar (\sum_{\alpha \in S_\ell} \gamma_\alpha \xi(w\alpha) - 1) - \sum_{\alpha \in S_\ell} \gamma_\alpha \eta_\alpha$ and γ_α are all non-negative or all non-positive. If $r_\ell = 0$ then $\xi|_{wS_\ell} \equiv 0$ and hence $wS_\ell \subset R_+$. Since $\theta \in wR_\ell$ it follows that $\theta = w\theta_\ell$ (cf. Lemma 3.8) which is a contradiction as we are in case (c) and not in case (a). If $r_\ell \neq 0$ then $R_\ell \cong A_{k_\ell}$, $\eta|_{R_\ell} = \text{const}$. The condition (3.31) implies

$$2\pi i p = \hbar \left(\sum_{\alpha \in S_\ell} \gamma_\alpha \xi(w\alpha) - 1 - \frac{r_\ell + 1}{k_\ell + 1} \sum_{\alpha \in S_\ell} \gamma_\alpha \right) \quad (3.40)$$

with $|\gamma_\alpha| \leq 1$. Since $\hbar \notin 2\pi i \mathbb{Q}$ the expression in the brackets vanishes and due to $|\sum_{\alpha \in S_\ell} \gamma_\alpha| = \sum_{\alpha \in S_\ell} |\gamma_\alpha| \leq k_\ell$ we have a contradiction with the condition (3.29). \square

Theorem 3.11 follows from Lemmas 3.12, 3.14 and 3.15.

Example 3.1. Let $R = A_n$, then $R_\ell \cong A_{k_\ell}$, $m - 1 + \sum_{\ell=1}^m k_\ell \leq n$ and $\eta_\alpha = \eta_1$ for all $\alpha \in R$. In this case the conditions (3.29) and (3.31) imply $r_1 = \dots = r_m$ and $k_1 = \dots = k_m$ such that $\text{GCD}(r_1 + 1, k_1 + 1) = 1$ and $(k_1 + 1)\eta_1 = (r_1 + 1)\hbar$. The corresponding invariant ideals (in the representation of DAHA of type GL_{n+1}) were constructed in [18].

4. Generalized Macdonald–Ruijsenaars systems

In this section we use the invariant ideals $I_{\mathcal{D}}$ found in Section 3 to obtain commutative algebras of difference operators acting in the space of functions defined on the affine planes. We work in the assumptions of Theorem 3.3, so that the ideal $I_{\mathcal{D}}$ is invariant and the quotient module is defined. The symmetrized Cherednik–Dunkl operators $m_\lambda(Y)$ considered in the quotient representation define a commutative algebra of difference operators. We consider explicit examples of operators arising this way.

4.1. The operation of restriction

In Section 2 we reviewed how the Macdonald–Ruijsenaars systems are obtained in terms of the polynomial representation of DAHAs. In this section we show how the generalized Macdonald–Ruijsenaars systems are obtained in terms of the representation

$$\rho_{\mathcal{D}} : \mathcal{H}_t \rightarrow \text{End}(\mathcal{F}/I_{\mathcal{D}}) \quad (4.1)$$

on the quotient module $\mathcal{F}/I_{\mathcal{D}}$. Thus $\rho_{\mathcal{D}}(a)(g) = a(f)|_{\mathcal{D}}$, where $a \in \mathcal{H}_t$, $f \in \mathcal{F}$ and $g = [f] = f|_{\mathcal{D}}$ is a class of the function f identified with its restriction on \mathcal{D} .

Consider the subspace $\mathcal{F}_0 \subset \mathcal{F}/I_{\mathcal{D}}$ of functions on \mathcal{D} which have W -invariant extension in V , that is $\mathcal{F}_0 = \{[f] \in \mathcal{F}/I_{\mathcal{D}} \mid f \in \mathcal{F}^W\}$. Since $\mathcal{D} \subset W\mathcal{D}_0$ the elements of \mathcal{F}_0 are determined by their values on the plane \mathcal{D}_0 and we can identify the space \mathcal{F}_0 with the space $\{f|_{\mathcal{D}_0} \mid f \in \mathcal{F}^W\}$ of functions on \mathcal{D}_0 .

Since $\mathbb{C}[Y]^W \mathcal{F}^W \subset \mathcal{F}^W$ the operators belonging to the algebra $\rho_{\mathcal{D}}(\mathbb{C}[Y]^W)$ preserve the space \mathcal{F}_0 . For any $a \in \mathbb{C}[Y]^W$ let

$$\text{res } \rho_{\mathcal{D}}(a) = \rho_{\mathcal{D}}(a)|_{\mathcal{F}_0}.$$

This operator as an operator on \mathcal{D}_0 is given more explicitly by the next theorem. Before its formulation we introduce some notations.

Let $\bar{V} = \{x \in V \mid (\alpha, x) = 0, \forall \alpha \in S_0\}$. Denote by $\bar{\lambda}$ the orthogonal projection of a vector $\lambda \in V$ onto the subspace \bar{V} and by \bar{P}^\vee the set of projections of the coweights: $\bar{P}^\vee = \{\bar{\lambda} \mid \lambda \in P^\vee\} \subset \bar{V}$. The shift operators $\tau(\bar{\lambda})$ preserve the affine plane \mathcal{D}_0 and therefore they act on functions ψ on this plane: $\tau(\bar{\lambda})\psi(x) = \psi(x - \hbar\bar{\lambda})$, where $x \in \mathcal{D}_0$. Let $\bar{\mathcal{M}}$ be the space of meromorphic functions on \mathcal{D}_0 which are restriction $\psi|_{\mathcal{D}_0}$, $\psi \in \mathcal{M}$. Define the algebra \mathbb{D} of difference operators on \mathcal{D}_0 by $\mathbb{D} = \bar{\mathcal{M}} \# \tau(\bar{P}^\vee)$. Note that any element $a \in \mathcal{H}_t$ in the representation (2.19) has the form

$$a = \sum_{\lambda \in P^\vee} \sum_{w \in W} g_{\lambda, w} \tau(\lambda) w, \quad (4.2)$$

where $g_{\lambda, w} \in \mathcal{M}$

Theorem 4.1. Let $a \in \mathbb{C}[Y]^W$ have the form (4.2). Then

$$\text{res } \rho_{\mathcal{D}}(a) = \sum_{\lambda' \in \overline{P^\vee}} \left(\sum_{w \in W} \sum_{\substack{\lambda \in P^\vee \\ \bar{\lambda} = \lambda'}} g_{\lambda, w} \right) \Big|_{\mathcal{D}_0} \tau(\lambda'). \quad (4.3)$$

Formula (4.3) defines the algebra homomorphism $\text{res} : \rho_{\mathcal{D}}(\mathbb{C}[Y]^W) \rightarrow \overline{\mathbb{D}}$.

Proof. Let us write $v \approx v'$ for $v, v' \in V$ if $wv - v' \in 2\pi i Q^\vee$ for some $w \in W$. Denote $P_\lambda^x = \{v \in P^\vee \mid x - \hbar v \approx x - \hbar \lambda\}$. Let $f_\lambda^{x_0}(x) \in \mathcal{F}^W$ be a function which equals 1 in the points $x \approx x_0 - \hbar \lambda$ and vanishes outside the small neighbourhoods of these points (such function exists if $x_0 \notin Z$). Note that for any $f \in \mathcal{F}^W$ the function $(af)(x)$ is smooth. Let $x_0 \in V \setminus Z$. Substituting $f = f_{\lambda_0}^{x_0}$ for $\lambda_0 \in P^\vee$ and taking the limit $x \rightarrow x_0$ in the formula

$$(af)(x) = \sum_{\lambda \in P^\vee} \sum_{w \in W} g_{\lambda, w}(x) f(x - \hbar \lambda) \quad (4.4)$$

we obtain $(af_{\lambda_0}^{x_0})(x_0) = \lim_{x \rightarrow x_0} G_{\lambda_0}^{x_0}(x)$, where $G_\lambda^{x_0}(x) = \sum_{v \in P_\lambda^{x_0}} \sum_{w \in W} g_{v, w}(x)$. So that the functions $G_\lambda^{x_0}(x)$ are defined in the point $x = x_0$ (have finite limit). Note also that $f|_{\mathcal{D}_0} = 0$ implies $(af)|_{\mathcal{D}_0} = 0$ since $aI_{\mathcal{D}} \subset I_{\mathcal{D}}$. Hence if $G_\lambda^{x_0}(x_0) \neq 0$ for some $x_0 \in \mathcal{D}_0 \setminus Z$, $\lambda \in P^\vee$ then there exists $w \in W$ such that $w(x_0 - \hbar \lambda) \in \mathcal{D}_0 + 2\pi i Q^\vee$.

Let the W -invariant function η satisfying the conditions $h_\eta^\ell = \hbar$ be generic. In particular we will assume that $Z \not\supset \mathcal{D}_0$. Let x_0 be a generic point of $\mathcal{D}_0 \setminus Z$ (that is a generic point of \mathcal{D}_0). Then it follows from $w(x_0 - \hbar \lambda) \in \mathcal{D}_0 + 2\pi i Q^\vee$ where $w \in W$ that $w_0(x_0 - \hbar \lambda) \in \mathcal{D}_0 + 2\pi i Q^\vee$ for some $w_0 \in W_0$. Indeed, for generic $x_0 \in \mathcal{D}_0$ the formula $w(x_0 - \hbar \lambda) \in \mathcal{D}_0 + 2\pi i Q^\vee$ implies $w^{-1}S_0 \subset V_0$ and hence $wV_0 = V_0$. Decomposing the element $w \in W$ as $w = w'w_0$ with $w' \in W_+$, $w_0 \in W_0$ (see Remark 3.1 and [17]) we obtain $w'V_0 = V_0$. Then by applying Proposition 3.2 we derive $w'\mathcal{D}_0 = \mathcal{D}_0$ and hence $w_0(x_0 - \hbar \lambda) \in \mathcal{D}_0 + 2\pi i Q^\vee$.

One can show (for generic function η) that $w_0(x_0 - \hbar \lambda) \in \mathcal{D}_0 + 2\pi i Q^\vee$ implies $x_0 - \hbar \lambda \approx x_0 - \hbar \bar{\lambda}$. Note that the right-hand side of (4.4) at $x = x_0$ has the form $(af)(x_0) = \sum_{\lambda \in P^\vee} \frac{1}{|P_\lambda^{x_0}|} G_\lambda^{x_0}(x_0) f(x_0 - \hbar \lambda)$. Since each term contributes only if $G_\lambda^{x_0}(x_0) \neq 0$ we can rewrite the expression as

$$(af)(x_0) = \sum_{\lambda \in P^\vee} \frac{1}{|P_\lambda^{x_0}|} G_\lambda^{x_0}(x_0) f(x_0 - \hbar \bar{\lambda}), \quad (4.5)$$

where we used the W -invariance and $2\pi i Q^\vee$ -periodicity of the function $f(x)$. One can prove that if $x_0 - \hbar \bar{\lambda} \approx x_0 - \hbar \bar{v}$ for some $\lambda, v \in P^\vee$ then $\bar{\lambda} = \bar{v}$. Thus we get $\bar{v} = \bar{\lambda}$ if $G_\lambda^{x_0}(x_0) \neq 0$ and $v \in P_\lambda^{x_0}$. Hence the formula (4.5) can be rearranged to

$$(af)(x_0) = \sum_{\lambda' \in \overline{P^\vee}} \left(\sum_{w \in W} \sum_{\substack{\lambda \in P^\vee \\ \bar{\lambda} = \lambda'}} g_{\lambda, w} \right) (x_0) f(x_0 - \hbar \lambda'). \quad (4.6)$$

We established the equality (4.6) for generic $x_0 \in \mathcal{D}_0$ and η . Since the left-hand side is defined for all $x_0 \in \mathcal{D}_0$ and η the equality is valid for all $x_0 \in \mathcal{D}_0$ and η . Since $(\text{res } \rho_{\mathcal{D}}(a)f|_{\mathcal{D}_0})(x_0) = (af)(x_0)$ for $f \in \mathcal{F}^W$ the equality (4.3) follows. Note that by putting $f = f_{\lambda'_0}^{x_0}$ where $\lambda'_0 \in \overline{P^\vee}$ in (4.6) we see that each function $(\sum_{w \in W} \sum_{\lambda \in P^\vee, \bar{\lambda} = \lambda'_0} g_{\lambda, w})(x_0)$ is defined for any admissible η and generic $x_0 \in \mathcal{D}_0$.

Notice that if $A \in \overline{\mathbb{D}}$ is such that $A\phi = 0$ for any $\phi \in \mathcal{F}_0$ then $A = 0$. Indeed consider the action of $A = \sum_{\lambda' \in \overline{P^\vee}} g_{\lambda'} \tau(\lambda')$ where $g_{\lambda'} \in \overline{\mathcal{M}}$ on $\phi = f_{\lambda'_0}^{x_0}|_{\mathcal{D}_0}$. Then one yields $g_{\lambda'_0}(x_0) = 0$ for generic $x_0 \in \mathcal{D}_0$ and all $\lambda'_0 \in \overline{P^\vee}$. It follows that the linear map $\text{res} : \rho_{\mathcal{D}}(\mathbb{C}[Y]^W) \rightarrow \overline{\mathbb{D}}$ defined by the formula (4.3) is an algebra homomorphism. \square

We are going to derive generalized Macdonald–Ruijsenaars operators using the operation $\text{res} \circ \rho_{\mathcal{D}}$ from Theorem 4.1. First it is useful to note that restrictions to the invariant functions and restrictions to the quotient module can be interchanged. More specifically, let $D \in \mathbb{D}$ be an operator of the form $D = \sum_{\lambda \in P^\vee} g_\lambda \tau(\lambda)$. Define the operator

$$\overline{D} = \sum_{\lambda' \in \overline{P^\vee}} \left(\sum_{\substack{\lambda \in P^\vee \\ \bar{\lambda} = \lambda'}} g_\lambda \right) \Big|_{\mathcal{D}_0} \tau(\lambda') \in \overline{\mathbb{D}} \quad (4.7)$$

in the case when for any $\lambda' \in \overline{P^\vee}$ the restricted function $(\sum_{\lambda \in P^\vee, \bar{\lambda} = \lambda'} g_\lambda)|_{\mathcal{D}_0}$ is well-defined meromorphic function on \mathcal{D}_0 .

Proposition 4.2. *The subspace $\overline{\mathfrak{R}} = \text{res} \circ \rho_{\mathcal{D}}(\mathbb{C}[Y]^W) \subset \overline{\mathbb{D}}$ is a commutative subalgebra in $\overline{\mathbb{D}}$. The map $\overline{\rho_D} : D \mapsto \overline{D}$ given by (4.7) is well defined on the subspace $\mathfrak{R} = \text{Res}(\mathbb{C}[Y]^W) \subset \mathbb{D}$. The algebra $\overline{\mathfrak{R}}$ coincides with the image of the map $\overline{\rho_D} : \mathfrak{R} \rightarrow \overline{\mathbb{D}}$, thus it is spanned by the elements $\overline{M_\lambda}$, $\lambda \in P^\vee$. The following diagram of surjective linear maps is commutative:*

$$\begin{array}{ccc} \mathbb{C}[Y]^W & \xrightarrow{\rho_D} & \rho_{\mathcal{D}}(\mathbb{C}[Y]^W) \\ \downarrow \text{Res} & & \downarrow \text{res} \\ \mathfrak{R} & \xrightarrow{\overline{\rho_D}} & \overline{\mathfrak{R}} \end{array} \quad (4.8)$$

and the map $\overline{\rho_D} : \mathfrak{R} \rightarrow \overline{\mathfrak{R}}$ is an algebra homomorphism.

Proof. The space $\overline{\mathfrak{R}}$ is a commutative algebra as an image of the homomorphism $\text{res} \circ \rho_{\mathcal{D}}$. The definition of the map $\text{res} : \rho_{\mathcal{D}}(\mathbb{C}[Y]^W) \rightarrow \overline{\mathbb{D}}$ implies that $\text{res} \circ \rho_{\mathcal{D}}(a) = \overline{\text{Res}(a)}$ for any $a \in \mathbb{C}[Y]^W$. Due to this formula and surjectivity of the map $\text{Res} : \mathbb{C}[Y]^W \rightarrow \mathfrak{R}$ the operator \overline{D} is defined for any $D \in \mathfrak{R}$. We also conclude that $\overline{\mathfrak{R}}$ coincides with image of the map $\overline{\rho_D} : \mathfrak{R} \rightarrow \overline{\mathbb{D}}$ and that the diagram (4.8) commutes. Since the operators M_λ form a basis in \mathfrak{R} their images $\overline{M_\lambda}$ span the space $\overline{\mathfrak{R}}$. The commutativity of the diagram implies in turn that $\overline{\rho_D} : \mathfrak{R} \rightarrow \overline{\mathfrak{R}}$ is a homomorphism: $\overline{a' \cdot b'} = \overline{\text{Res}(a) \cdot \text{Res}(b)} = \overline{\text{Res}(a \cdot b)} = \text{res} \circ \rho_{\mathcal{D}}(a \cdot b) = \text{res} \circ \rho_{\mathcal{D}}(a) \cdot \text{res} \circ \rho_{\mathcal{D}}(b) = \overline{a'} \cdot \overline{b'}$, where $a, b \in \mathbb{C}[Y]^W$ such that $a' = \text{Res}(a)$ and $b' = \text{Res}(b)$. \square

We will say that the image \overline{D} of $D \in \mathfrak{R}$ under $\overline{\rho_D}$ is the *restriction* of the operator D onto \mathcal{D}_0 . The commutative subalgebra $\overline{\mathfrak{R}} \subset \overline{\mathbb{D}}$ is the *generalized Macdonald–Ruijsenaars system*. The restrictions $\overline{M_{-b_r}}$ and $\overline{M_{-\theta^\vee}}$ of the Macdonald operators M_{-b_r} and $M_{-\theta^\vee}$ are the *generalized Macdonald–Ruijsenaars operators*.

4.2. Integrability of generalized Macdonald–Ruijsenaars systems

Let $\bar{n} = \dim \mathcal{D}_0 = n - |S_0|$, where $|S_0|$ is a number of elements in the subset $S_0 \subset \Delta$. The set of indexes $J = \{j \in \mathbb{Z}_{>0} \mid \alpha_j \notin S_0\}$ has size \bar{n} , let $J = \{j_1, \dots, j_{\bar{n}}\}$. If $j \in J$ then the corresponding

fundamental coweight belongs to the subspace \bar{V} , that is $\bar{b}_j = b_j$, $b_j \in \bar{P}^\vee$ and $\tau(\pm b_j)$ preserve the plane \mathcal{D}_0 .

Theorem 4.3. *The operators $\overline{M_{-b_{j_1}}}, \dots, \overline{M_{-b_{j_n}}}$ are algebraically independent elements of $\bar{\mathfrak{A}}$.*

Proof. By the formula (2.31) with $\lambda = -b_j$ we have

$$M_{-b_j} = g^{-b_j}(x)\tau(-b_j) + \sum_{v < -b_j} g_v^{-b_j}(x)\tau(v), \quad (4.9)$$

where $g_v^{-b_j} \in \mathcal{M}$ and $g^{-b_j}(x) = \prod_{\alpha \in \hat{R}_{-b_j}} \frac{t_\alpha e^{\alpha(x)} - t_\alpha^{-1}}{e^{\alpha(x)} - 1}$.

Let $j \in J$. Firstly note that $v < -b_j$ implies $\bar{v} \neq -b_j$. Indeed, if a weight $v \in P^\vee$ satisfies $v < -b_j$ then it follows the corresponding inequality between the lengths: $|v| \leq |-b_j|$. If $\bar{v} = v$ then $\bar{v} = v \neq -b_j$. If $\bar{v} \neq v$ then one has $|\bar{v}| < |v|$ hence $|\bar{v}| < |-b_j|$ and therefore $\bar{v} \neq -b_j$. Taking into account that $\bar{b}_j = b_j$ one yields

$$\overline{M_{-b_j}} = g^{-b_j}|_{\mathcal{D}_0} \tau(-b_j) + \sum_{\substack{v' \in \bar{P}^\vee \\ v' \neq -b_j}} \left(\sum_{\substack{v < -b_j \\ \bar{v} = v'}} g_v^{-b_j} \right) \Big|_{\mathcal{D}_0} \tau(v'). \quad (4.10)$$

Let us prove that the function $g^{-b_j}|_{\mathcal{D}_0}$ does not vanish identically. Indeed, otherwise there exists an affine root $\alpha = [\alpha^{(1)}, \alpha^{(0)}] \in \hat{R}_{-b_j}$ such that $e^{\alpha(x)} \equiv e^{(\alpha^{(1)}, x) + \alpha^{(0)}} = t_\alpha^{-2}$ for all $x \in \mathcal{D}_0$. This implies that $\alpha^{(1)} \in R$ is a linear combination of the elements of S_0 and, consequently, $(\alpha^{(1)}, b_j) = 0$. But on the other hand

$$\hat{R}_{-b_j} = (\tau(-b_j)\hat{R}_+) \cap \hat{R}_- = \{[\beta, \hbar k] \mid \beta \in R_-, k \in \mathbb{Z}, (\beta, b_j) < k \leq 0\},$$

which implies $(\alpha^{(1)}, b_j) < 0$. Hence $g^{-b_j}|_{\mathcal{D}_0} \neq 0$.

Let us introduce the order on the set \bar{P}^\vee : we will write $v' \succ \lambda'$ for $v', \lambda' \in \bar{P}^\vee$ if $v' \neq \lambda'$ and $v' - \lambda' = \sum_{i=1}^n \gamma_i \bar{\alpha}_i^\vee$ for some $\gamma_i \in \mathbb{Z}_{\geq 0}$. One can check that this is a partial order on \bar{P}^\vee . The ordering $v < -b_j$ for $v \in P^\vee$ implies $v > -b_j$, $\bar{v} \neq -b_j$ and hence $\bar{v} \succ -b_j$. So denoting $\bar{g}_{v'}^{-b_j} = \sum_{v < -b_j, \bar{v} = v'} g_v^{-b_j} \in \bar{\mathcal{M}}$ for $v' \neq -b_j$ we obtain

$$\overline{M_{-b_j}} = g^{-b_j}|_{\mathcal{D}_0} \tau(-b_j) + \sum_{\substack{v' \in \bar{P}^\vee \\ v' \succ -b_j}} \bar{g}_{v'}^{-b_j} \tau(v'), \quad (4.11)$$

where $g^{-b_j}|_{\mathcal{D}_0} \neq 0$. Let $m_1, \dots, m_{\bar{n}} \in \mathbb{Z}_{\geq 0}$ and $\lambda' = -\sum_{l=1}^{\bar{n}} m_l b_{j_l} \in \bar{P}_-^\vee$, then

$$\prod_{l=1}^{\bar{n}} (\overline{M_{-b_{j_l}}})^{m_l} = g^{m_1, \dots, m_{\bar{n}}} \tau(\lambda') + \sum_{\substack{v' \in \bar{P}^\vee \\ v' \succ \lambda'}} g_{v'}^{m_1, \dots, m_{\bar{n}}} \tau(v'), \quad (4.12)$$

where $g^{m_1, \dots, m_{\bar{n}}}, \bar{g}_{v'}^{m_1, \dots, m_{\bar{n}}} \in \bar{\mathcal{M}}$ and $g^{m_1, \dots, m_{\bar{n}}} \neq 0$. Since the operators (4.12) are linearly independent the elements $\overline{M_{-b_{j_1}}}, \dots, \overline{M_{-b_{j_n}}}$ are algebraically independent. \square

Now we introduce some notations. Let f be a function of several variables depending, in particular, on a complex variable z . Then we denote by \mathcal{T}_z^a the operator acting on f by the shift:

$$(\mathcal{T}_z^a f)(\dots, z, \dots) = f(\dots, z + a, \dots), \quad (4.13)$$

where $a \in \mathbb{C}$. Let $x_i = (e_i, x) + c_i$ for $i = 1, \dots, N$, where $\{e_i\}$ is a set of pairwise orthogonal vectors in \mathbb{C}^N and $c_i \in \mathbb{C}$. Then we have $\mathcal{T}_{x_i}^{hK_i} = \tau(-e_i)$ on the functions $f(x) = f(x_1, \dots, x_N)$, where $K_i = (e_i, e_i)$.

4.3. A_n type examples

The root system $R = A_n$ is defined as $\{e_i - e_j \mid i, j = 1, \dots, n+1, i \neq j\}$ considered in the space $V = \{x = \sum_{i=1}^{n+1} x_i e_i \in \mathbb{C}^{n+1} \mid \sum_{i=1}^{n+1} x_i = 0\}$, where $\{e_i\}$ is the standard orthonormal basis in \mathbb{C}^{n+1} . The simple roots are $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \dots, \alpha_n = e_n - e_{n+1}$ and the maximal root is $\theta = e_1 - e_{n+1}$. All fundamental coweights $b_r, r = 1, \dots, n$ are minuscule. Any \widehat{W} -invariant function $t: \widehat{R} \rightarrow \mathbb{C} \setminus \{0\}$ is constant.

We will consider functions f on V as functions of $n+1$ variables satisfying the condition $\prod_{i=1}^{n+1} \mathcal{T}_{x_i}^a f(x_1, \dots, x_{n+1}) = f(x_1, \dots, x_{n+1}), \forall a \in \mathbb{C}$. Then we have $\mathcal{T}_{x_i}^h f = \tau(-e_i) f = \tau(-\hat{e}_i) f$, where \hat{e}_i is the orthogonal projection of the vector e_i to the subspace $V \subset \mathbb{C}^{n+1}$.

Taking the coweight $b_1 = \hat{e}_1$ in (2.25) we obtain the operator

$$M_{-b_1} = \sum_{i=1}^{n+1} \prod_{\substack{j=1 \\ j \neq i}}^{n+1} \frac{te^{x_j - x_i} - t^{-1}}{e^{x_j - x_i} - 1} \mathcal{T}_{x_i}^h. \quad (4.14)$$

Any root subsystem R_0 has irreducible components R_ℓ of type A only in its decomposition $R_0 = \bigsqcup_{\ell=1}^m R_\ell$. The invariant function $\eta = \text{const}$. The generalized Coxeter number is $h_\eta^\ell = k\eta$ for $R_\ell \cong A_{k-1}$. Thus the conditions (3.13) lead to $R_1 \cong R_2 \cong \dots \cong R_m \cong A_{k-1}$ and $\hbar = k\eta$ for some $k \in \mathbb{Z}_{\geq 2}$. Let us choose

$$S_\ell = \{e_{N_1+k(\ell-1)+1} - e_{N_1+k(\ell-1)+2}, \dots, e_{N_1+k\ell-1} - e_{N_1+k\ell}\}, \quad \ell = 1, \dots, N_2, \quad (4.15)$$

where $N_1 = n+1 - mk$ and $N_2 = m$. Introduce new variables

$$y_\ell = \frac{1}{k} \sum_{s=0}^{k-1} (x_{N_1+k\ell-s} - s\eta) = \frac{1}{k} \sum_{s=0}^{k-1} x_{N_1+k\ell-s} - \frac{k-1}{2} \eta, \quad \ell = 1, \dots, N_2. \quad (4.16)$$

Note that $x_{N_1+k\ell-s}|_{\mathcal{D}_0} = y_\ell|_{\mathcal{D}_0} + s\eta$ for all $s = 0, \dots, k-1$. The values of the variables x_i, y_ℓ with $i = 1, \dots, N_1, \ell = 1, \dots, N_2$ satisfying $\sum_{i=1}^{N_1} x_i + k \sum_{\ell=1}^{N_2} y_\ell = -k(k-1)\eta/2$ uniquely determine a point on the affine plane \mathcal{D}_0 . The functions $f: \mathcal{D}_0 \rightarrow \mathbb{C}$ are functions $f(x_1, \dots, x_{N_1}, y_1, \dots, y_{N_2})$ satisfying $\prod_{i=1}^{N_1} \mathcal{T}_{x_i}^a \prod_{\ell=1}^{N_2} \mathcal{T}_{y_\ell}^{a/k} f = f, \forall a \in \mathbb{C}$.

Proposition 4.4. Let $t = e^{\eta/2}$. The restriction of the operator (4.14) onto \mathcal{D}_0 takes the form

$$\begin{aligned} \overline{M}_{-b_1} &= \sum_{i=1}^{N_1} \prod_{\substack{j=1 \\ j \neq i}}^{N_1} \frac{te^{x_j - x_i} - t^{-1}}{e^{x_j - x_i} - 1} \prod_{\ell'=1}^{N_2} \frac{qe^{y_{\ell'} - x_i} - q^{-1}}{e^{y_{\ell'} - x_i} - 1} \mathcal{T}_{x_i}^h \\ &\quad + \frac{q - q^{-1}}{t - t^{-1}} \sum_{\ell=1}^{N_2} \prod_{j=1}^{N_1} \frac{te^{x_j - y_\ell} - t^{-1}}{e^{x_j - y_\ell} - 1} \prod_{\substack{\ell'=1 \\ \ell' \neq \ell}}^{N_2} \frac{qe^{y_{\ell'} - y_\ell} - q^{-1}}{e^{y_{\ell'} - y_\ell} - 1} \mathcal{T}_{y_\ell}^\eta. \end{aligned} \quad (4.17)$$

Proof. The coefficient at $\mathcal{T}_{x_i}^h$ in M_{-b_1} has non-zero restriction on \mathcal{D}_0 only if $1 \leq i \leq N_1$ or $i = N_1 + k\ell$ for some $\ell = 1, \dots, N_2$. These restrictions can be calculated explicitly using the formulae (valid on \mathcal{D}_0)

$$\prod_{s=0}^{k-1} g(x_{N_1+k\ell'-s} - a; t, 1) = g(y_{\ell'} - a; t^k, 1),$$

$$\prod_{s=1}^{k-1} g(x_{N_1+k\ell-s} - x_{N_1+k\ell}; t, 1) = g(0; t^k, t) = \frac{t^k - t^{-k}}{t - t^{-1}},$$

where $a \in \mathbb{C}$ and $g(z; b, c) = \frac{be^z - b^{-1}}{ce^z - c^{-1}}$. Further since $\bar{e}_i = e_i$ for $i = 1, \dots, N_1$ the restriction of $\mathcal{T}_{x_i}^h = \tau(-e_i)$ in this case coincides with $\mathcal{T}_{x_i}^h$. For $i = N_1 + k\ell$ the restriction of $\mathcal{T}_{x_i}^h = \tau(-e_i)$ is equal to $\mathcal{T}_{y_\ell}^{h/k}$ due to $\bar{e}_i = \frac{1}{k} \sum_{s=0}^{k-1} e_{N_1+k\ell-s}$ and $y_\ell = (\bar{e}_i, x) - (k-1)\eta/2$ (see the end of Section 4.2). Taking into account the condition $\hbar = k\eta$ and its exponential form $q = t^k$ we can write the restriction of (4.14) in the form (4.17). \square

Note that the operator (4.17) does not effectively depend on k , it depends on N_1, N_2, \hbar, η only, where it is assumed that $\hbar/\eta \in \mathbb{Z}_{>0}$. The operator (4.17) was previously obtained and investigated in [29] for general parameters \hbar and η and earlier in [2] in the case $N_2 = 1$.

The formula (2.25) gives the explicit form of n Macdonald operators

$$M_{-b_r} = \sum_{\substack{I \subset \{1, \dots, n+1\} \\ |I|=r}} \prod_{\substack{i \in I \\ j \notin I}} \frac{te^{x_j - x_i} - t^{-1}}{e^{x_j - x_i} - 1} \prod_{i \in I} \mathcal{T}_{x_i}^h, \quad r = 1, \dots, n, \quad (4.18)$$

where $|I|$ is the number of elements in the set $I \subset \{1, \dots, n+1\}$. Similar to the proof of Proposition 4.4 we calculate the restrictions of the operators (4.18) onto \mathcal{D}_0 . We obtain

$$\begin{aligned} \overline{M_{-b_r}} = & \sum_{\substack{I \subset \{1, \dots, N_1\} \\ m_\ell \in \mathbb{Z}_{\geq 0} \\ |I| + \sum_{\ell=1}^{N_2} m_\ell = r}} \prod_{\substack{i, j=1 \\ i \in I \\ j \notin I}}^{N_1} \frac{te^{x_j - x_i} - t^{-1}}{e^{x_j - x_i} - 1} \prod_{\ell, \ell'=1}^{N_2} \prod_{s=0}^{m_{\ell'}-1} \frac{qt^{-s} e^{y_{\ell'} - y_\ell} - q^{-1} t^s}{t^{m_{\ell'}-s} e^{y_{\ell'} - y_\ell} - t^{s-m_{\ell'}}} \\ & \times \prod_{\substack{\ell=1 \\ i \in I}}^{N_2} \frac{qe^{y_\ell - x_i} - q^{-1}}{t^{m_\ell} e^{y_\ell - x_i} - t^{-m_\ell}} \prod_{\substack{\ell=1 \\ j \notin I}}^{N_2} \frac{te^{x_j - y_\ell} - t^{-1}}{t^{1-m_\ell} e^{x_j - y_\ell} - t^{m_\ell-1}} \prod_{i \in I} \mathcal{T}_{x_i}^h \prod_{\ell=1}^{N_2} \mathcal{T}_{y_\ell}^{m_\ell \eta}, \end{aligned} \quad (4.19)$$

where $q = e^{\hbar/2}$, $t = e^{\eta/2}$. In the case $r = 1$ the operator (4.19) takes the form (4.17).

Theorem 4.5. For any values of the parameters \hbar and η the operators $\overline{M_{-b_r}}$ given by the formula (4.19), $r \in \mathbb{Z}_{>0}$, pairwise commute. The operators $\overline{M_{-b_r}}$ with $r = 1, \dots, N_1 + N_2 - 1$ are algebraically independent at generic values of \hbar and η .

Proof. For any $r, r' \in \mathbb{Z}_{>0}$ the commutator has the form

$$[\overline{M_{-b_r}}, \overline{M_{-b_{r'}}}] = \sum_{\substack{k_1, \dots, k_{N_1} \in \mathbb{Z}_{\geq 0} \\ m_1, \dots, m_{N_2} \in \mathbb{Z}_{\geq 0}}} C_{m_1, \dots, m_{N_2}}^{k_1, \dots, k_{N_1}}(x, y; q, t) \prod_{i=1}^{N_1} \mathcal{T}_{x_i}^{k_i \hbar} \prod_{\ell=1}^{N_2} \mathcal{T}_{y_\ell}^{m_\ell \eta}, \quad (4.20)$$

where the coefficients $C_{m_1, \dots, m_{N_2}}^{k_1, \dots, k_{N_1}}(x, y; q, t)$ are rational functions of $q = e^{\hbar/2}$ and $t = e^{\eta/2}$. It follows from Proposition 4.2 that these functions equal zero if $\hbar, \eta \in \mathbb{C} \setminus \pi i \mathbb{Q}$ such that $\hbar/\eta \in \mathbb{Z}_{>1}$ and $N_1 + \frac{\hbar}{\eta} N_2 > \max(r, r')$. Thus the coefficients vanish for any parameters \hbar and η , and the operators commute.

The second part follows from the fact that the operators $\overline{M_{-b_r}}$ at $\hbar = \eta$ are specializations of the Macdonald operators M_{-b_r} with $n = N_1 + N_2 - 1$, which are algebraically independent. \square

4.4. B_n type examples

The root system $R = B_n$ is defined as the set $\{\epsilon e_i + \epsilon' e_j \mid 1 \leq i < j \leq n, \epsilon, \epsilon' = \pm 1\} \cup \{\epsilon e_i \mid 1 \leq i \leq n, \epsilon = \pm 1\}$. The simple roots are $\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \dots, \alpha_{n-1} = e_{n-1} - e_n, \alpha_n = e_n$. The maximal root is $\theta = e_1 + e_2$. The group \widehat{W} has two orbits on \widehat{R} so the function $t : \widehat{R} \rightarrow \mathbb{C} \setminus \{0\}$ is determined by the values $t_1 = t_{\pm e_i \pm e_j}$ and $t_n = t_{\pm e_i}$. There is a unique minuscule coweight $b_1 = e_1$. The corresponding Macdonald operator is

$$\begin{aligned} M_{-e_1} = & \sum_{i=1}^n \left(\prod_{\substack{j=1 \\ j \neq i}}^n \frac{(t_1 e^{x_i - x_j} - t_1^{-1})(t_1 e^{x_i + x_j} - t_1^{-1})}{(e^{x_i - x_j} - 1)(e^{x_i + x_j} - 1)} \frac{t_n e^{x_i} - t_n^{-1}}{e^{x_i} - 1} \mathcal{T}_{x_i}^{-\hbar} \right. \\ & \left. + \prod_{\substack{j=1 \\ j \neq i}}^n \frac{(t_1 e^{x_j - x_i} - t_1^{-1})(t_1 e^{-x_i - x_j} - t_1^{-1})}{(e^{x_j - x_i} - 1)(e^{-x_i - x_j} - 1)} \frac{t_n e^{-x_i} - t_n^{-1}}{e^{-x_i} - 1} \mathcal{T}_{x_i}^{\hbar} \right), \end{aligned} \quad (4.21)$$

where $x_i = (e_i, x)$ are coordinates on V .

The invariant function η satisfies $\eta_1 = \eta_2 = \dots = \eta_{n-1}$ and $\eta_\theta = \eta_1$. The generalized Coxeter number is $h_\eta^\ell = k\eta_1$ for $R_\ell \cong A_{k-1}$ and $h_\eta^\ell = 2(k-1)\eta_1 + 2\eta_n$ if $R_\ell \cong B_k$.

We have two cases $\alpha_n \notin S_0$ and $\alpha_n \in S_0$ depending on the choice of $S_0 \subset \Delta$. Consider first the case $\alpha_n \notin S_0$. All the irreducible components R_ℓ are of type A . Then the conditions $h_\eta^\ell = \hbar$ imply $\hbar = k\eta_1$ for some $k \in \mathbb{Z}_{>0}$ and $R_\ell \cong A_{k-1}$ for all ℓ . Hence $q = t_1^k$ and we can choose

$$S_\ell = \{e_{N_1+k(\ell-1)+1} - e_{N_1+k(\ell-1)+2}, \dots, e_{N_1+k\ell-1} - e_{N_1+k\ell}\}, \quad \ell = 1, \dots, N_2, \quad (4.22)$$

where $N_1 = n - mk$ and $N_2 = m$. Define new variables

$$y_\ell = \frac{1}{k} \sum_{s=0}^{k-1} x_{N_1+k\ell-s} = \frac{k-1}{2} \eta_1 + \frac{1}{k} \sum_{s=0}^{k-1} (x_{N_1+k\ell-s} - s\eta_1), \quad \ell = 1, \dots, N_2. \quad (4.23)$$

Note that $x_{N_1+k\ell-s} |_{\mathcal{D}_0} = y_\ell |_{\mathcal{D}_0} - (k-1)\eta_1/2 + s\eta_1$ for all $s = 0, \dots, k-1$. The variables x_i, y_ℓ with $i = 1, \dots, N_1, \ell = 1, \dots, N_2$ form a system of orthogonal coordinates on the affine plane \mathcal{D}_0 .

Proposition 4.6. *Let $t_1 = e^{\eta_1/2}$. In the case $\alpha_n \notin S_0$ the restriction of the operator (4.21) onto \mathcal{D}_0 takes the form*

$$\overline{M_{-e_1}} = \sum_{\substack{i=1 \\ \epsilon=\pm 1}}^{N_1} A_i^\epsilon(x, y) \mathcal{T}_{x_i}^{-\epsilon\hbar} + \frac{q - q^{-1}}{t_1 - t_1^{-1}} \sum_{\substack{\ell=1 \\ \epsilon=\pm 1}}^{N_2} B_\ell^\epsilon(x, y) \mathcal{T}_{y_\ell}^{-\epsilon\eta_1}, \quad (4.24)$$

where

$$\begin{aligned}
A_i^\epsilon(x, y) &= \prod_{\substack{j=1 \\ j \neq i}}^{N_1} (g(\epsilon x_i - x_j; t_1, 1) g(\epsilon x_i + x_j; t_1, 1)) \\
&\quad \times \prod_{\ell'=1}^{N_2} (g(\epsilon x_i - y_{\ell'}; q^{1/2} t_1^{1/2}, q^{-1/2} t_1^{1/2}) g(\epsilon x_i + y_{\ell'}; q^{1/2} t_1^{1/2}, q^{-1/2} t_1^{1/2})) \\
&\quad \times g(\epsilon x_i; t_n, 1),
\end{aligned} \tag{4.25}$$

$$\begin{aligned}
B_\ell^\epsilon(x, y) &= \prod_{j=1}^{N_1} (g(\epsilon y_\ell - x_j; q^{1/2} t_1^{1/2}, q^{1/2} t_1^{-1/2}) g(\epsilon y_\ell + x_j; q^{1/2} t_1^{1/2}, q^{1/2} t_1^{-1/2})) \\
&\quad \times \prod_{\substack{\ell'=1 \\ \ell' \neq \ell}}^{N_2} (g(\epsilon y_\ell - y_{\ell'}; q, 1) g(\epsilon y_\ell + y_{\ell'}; q, 1)) g(\epsilon y_\ell; q^{1/2} t_1^{-1/2} t_n, q^{1/2} t_1^{-1/2}) \\
&\quad \times g(2\epsilon y_\ell; q t_1^{-1}, 1),
\end{aligned} \tag{4.26}$$

and

$$g(z; a, b) = \frac{ae^z - a^{-1}}{be^z - b^{-1}}, \quad a, b \in \mathbb{C}. \tag{4.27}$$

Proof. The coefficient at $\mathcal{T}_{x_i}^{-\epsilon h}$ in M_{-e_1} has non-zero restriction on \mathcal{D}_0 only if $1 \leq i \leq N_1$ or $i = N_1 + k\ell - (1 + \epsilon)(k - 1)/2$ for some $\ell = 1, \dots, N_2$. To calculate these coefficients we first use

$$\prod_{s=0}^{k-1} g(a \pm x_{N_1+k\ell'-s}; t_1, 1) = g(a \pm y_{\ell'}; q^{1/2} t_1^{1/2}, q^{-1/2} t_1^{1/2}), \tag{4.28}$$

$$\begin{aligned}
&\prod_{\substack{s=0 \\ s \neq N_1+k\ell-i}}^{k-1} g(\epsilon x_i - x_{N_1+k\ell-s}; t_1, 1) g(\epsilon x_i + x_{N_1+k\ell-s}; t_1, 1) \\
&= g(0; q, t) g(2\epsilon y_\ell; q t_1^{-1}, 1),
\end{aligned} \tag{4.29}$$

where $i = N_1 + k\ell - (1 + \epsilon)(k - 1)/2$. Further, if $i = N_1 + k\ell - (1 + \epsilon)(k - 1)/2$ then we have $\epsilon x_i = \epsilon(y_\ell - (k - 1)\eta_1/2 + (1 + \epsilon)(k - 1)\eta_1/2) = \epsilon y_\ell + (k - 1)\eta_1/2$ and hence

$$g(\epsilon x_i + a; b, c) = g(\epsilon y_\ell + a; bq^{1/2} t_1^{-1/2}, cq^{1/2} t_1^{-1/2}) \tag{4.30}$$

for any $b, c \neq 0$. Thus we obtain (4.24). \square

If $\alpha_n \in S_0$ then we suppose that the numeration of the components S_ℓ is such that $\alpha_n \in S_m$. Then $R_\ell \cong A_{k-1}$ for $\ell = 1, \dots, m - 1$ and $R_m \cong B_{\tilde{k}}$, where $k = |S_1| + 1$ and $\tilde{k} = |S_m|$. Put $N_1 = n - (m - 1)k - \tilde{k}$, $N_2 = m - 1$, and let S_1, \dots, S_{m-1} have the form (4.22), while

$$S_m = \{e_{n-\tilde{k}+1} - e_{n-\tilde{k}+2}, e_{n-\tilde{k}+2} - e_{n-\tilde{k}+3}, \dots, e_{n-1} - e_n, e_n\}. \tag{4.31}$$

The variables x_1, \dots, x_{N_1} together with (4.23) are coordinates on \mathcal{D}_0 .

Proposition 4.7. Let $t_1 = e^{\eta_1/2}$ and $t_n = e^{\eta_n/2}$. In the case $\alpha_n \in S_0$ the restriction of the operator (4.21) onto \mathcal{D}_0 takes the form

$$\overline{M_{-e_1}} = \sum_{\substack{i=1 \\ \epsilon=\pm 1}}^{N_1} \tilde{A}_i^\epsilon(x, y) \mathcal{T}_{x_i}^{-\epsilon\hbar} + \frac{q - q^{-1}}{t_1 - t_1^{-1}} \sum_{\substack{\ell=1 \\ \epsilon=\pm 1}}^{N_2} \tilde{B}_\ell^\epsilon(x, y) \mathcal{T}_{y_\ell}^{-\epsilon\eta_1} + \tilde{C}(x, y), \quad (4.32)$$

where

$$\begin{aligned} \tilde{A}_i^\epsilon(x, y) &= g(\epsilon x_i; t_1 t_n^{-1}, q^{-1/2}) g(\epsilon x_i; q^{1/2} t_1, 1) \prod_{\substack{j=1 \\ j \neq i}}^{N_1} g(\epsilon x_i - x_j; t_1, 1) g(\epsilon x_i + x_j; t_1, 1) \\ &\quad \times \prod_{\ell'=1}^{N_2} g(\epsilon x_i - y_{\ell'}; q^{1/2} t_1^{1/2}, q^{-1/2} t_1^{1/2}) \\ &\quad \times g(\epsilon x_i + y_{\ell'}; q^{1/2} t_1^{1/2}, q^{-1/2} t_1^{1/2}), \end{aligned} \quad (4.33)$$

$$\begin{aligned} \tilde{B}_\ell^\epsilon(x, y) &= g(\epsilon y_\ell; q^{1/2} t_1^{1/2} t_n^{-1}, t_1^{-1/2}) g(\epsilon y_\ell; q t_1^{1/2}, q^{1/2} t_1^{-1/2}) g(2\epsilon y_\ell; q t_1^{-1}, 1) \\ &\quad \times \prod_{j=1}^{N_1} g(\epsilon y_\ell - x_j; q^{1/2} t_1^{1/2}, q^{1/2} t_1^{-1/2}) g(\epsilon y_\ell + x_j; q^{1/2} t_1^{1/2}, q^{1/2} t_1^{-1/2}) \\ &\quad \times \prod_{\substack{\ell'=1 \\ \ell' \neq \ell}}^{N_2} g(\epsilon y_\ell - y_{\ell'}; q, 1) g(\epsilon y_\ell + y_{\ell'}; q, 1), \end{aligned} \quad (4.34)$$

$$\begin{aligned} \tilde{C}(x, y) &= \frac{(q^{1/2} t_1 t_n^{-1} - q^{-1/2} t_1^{-1} t_n)(q^{1/2} + q^{-1/2})}{t_1 - t_1^{-1}} \\ &\quad \times \prod_{j=1}^{N_1} g(-x_j; q^{1/2} t_1, q^{1/2}) g(x_j; q^{1/2} t_1, q^{1/2}) \\ &\quad \times \prod_{\ell'=1}^{N_2} g(-y_{\ell'}; q t_1^{1/2}, t_1^{1/2}) g(y_{\ell'}; q t_1^{1/2}, t_1^{1/2}), \end{aligned} \quad (4.35)$$

and $g(z; a, b)$ is defined by (4.27).

Proof. Note that $x_{n-s} = \eta_n + s\eta_1$ for all $s = 0, \dots, \tilde{k} - 1$. So the functional coefficient at $\mathcal{T}_{x_i}^{-\epsilon\hbar}$ has non-zero restriction on \mathcal{D}_0 only if $1 \leq i \leq N_1$ or $i = N_1 + k\ell - (1 + \epsilon)(k - 1)/2$ for some $\ell = 1, \dots, N_2$ or $i - n + \tilde{k} = \epsilon = +1$. To calculate the coefficients for the first two cases we use the formulae (4.28), (4.29), (4.30) and the formula

$$g(a; t_n, 1) \prod_{s=0}^{\tilde{k}-1} g(a - x_{n-s}; t_1, 1) g(a + x_{n-s}; t_1, 1) = g(a; t_1 t_n^{-1}; q^{1/2}) g(a; q^{1/2} t_1, 1).$$

The restriction of $\mathcal{T}_{x_{n-\tilde{k}+1}}^{-\hbar}$ yields the identity operator 1. The corresponding coefficient is calculated using

$$g(x_{n-\tilde{k}+1} + a; b, c) = g(a; bt_1^{\tilde{k}-1}t_n, ct_1^{\tilde{k}-1}t_n) = g(a; bq^{1/2}, cq^{1/2}),$$

$$g(x_i; t_n, 1) \prod_{s=0}^{\tilde{k}-2} g(x_i - x_{n-s}; t_1, 1)g(x_i + x_{n-s}; t_1, 1) = g(0; q^{1/2}t_1t_n^{-1}, t_1)g(0; q, q^{1/2}),$$

where $i = n - \tilde{k} + 1$ and we used the condition $2(\tilde{k} - 1)\eta_1 + 2\eta_n = \hbar$ in the form $t_1^{\tilde{k}-1}t_n = \pm q^{1/2}$. \square

Remark 4.1. The operator (4.32) can be rewritten in the form

$$\overline{M_{-e_1}} = \sum_{\substack{i=1 \\ \epsilon=\pm 1}}^{N_1} \tilde{A}_i^\epsilon(x, y)(\mathcal{T}_{x_i}^{-\epsilon\hbar} - 1) + \frac{q - q^{-1}}{t_1 - t_1^{-1}} \sum_{\substack{\ell=1 \\ \epsilon=\pm 1}}^{N_2} \tilde{B}_\ell^\epsilon(x, y)(\mathcal{T}_{y_\ell}^{-\epsilon\eta_1} - 1) + m_{-e_1, t}, \quad (4.36)$$

where $m_{-e_1, t} = \frac{t_1^n - t_1^{-n}}{t_1 - t_1^{-1}}(t_n t_1^{n-1} + t_n^{-1} t_1^{1-n})$.

Remark 4.2. Rational degeneration of the operators (4.24) and (4.32) can be obtained by substitution $x_i \rightarrow x_i/\omega$, $y_i \rightarrow y_i/\omega$, $\eta_\alpha \rightarrow \eta_\alpha/\omega$, $\hbar \rightarrow \hbar/\omega$ and taking the limit $\omega \rightarrow \infty$. This gives particular case of the operator considered by Sergeev and Veselov in [30, formula (94)], see also [12].

4.5. An example from root system F_4

The root system $R = F_4$ is given in $V = \mathbb{C}^4$ by

$$R = \{\epsilon e_i \mid 1 \leq i \leq 4, \epsilon = \pm 1\} \cup \{\epsilon_1 e_i + \epsilon_2 e_j \mid 1 \leq i < j \leq 4, \epsilon_1, \epsilon_2 = \pm 1\} \cup \left\{ \frac{1}{2} \sum_{k=1}^4 v_k e_k \mid v_1, \dots, v_4 = \pm 1 \right\}. \quad (4.37)$$

The simple roots are $\alpha_1 = e_2 - e_3$, $\alpha_2 = e_3 - e_4$, $\alpha_3 = e_4$ and $\alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$. The maximal root is $\theta = e_1 + e_2$. The corresponding Weyl group W has two orbits on R . There are no minuscule coweights in this case. The W -invariant function η satisfies $\eta_1 = \eta_2$ and $\eta_3 = \eta_4$. We impose $t_0 = t_1 = t_2 = e^{\eta_1/2}$ and $t_3 = t_4 = e^{\eta_3/2}$. The Macdonald operator (2.27) takes the form

$$M_{-\theta^\vee} = \sum_{\substack{1 \leq i < j \leq 4 \\ \epsilon_1, \epsilon_2 = \pm 1}} g(\epsilon_1 x_i; t_3, 1)g(\epsilon_2 x_j; t_3, 1) \\ \times \prod_{\substack{k \neq i, j \\ \epsilon_3 = \pm 1}} (g(\epsilon_1 x_i + \epsilon_3 x_k; t_1, 1)g(\epsilon_2 x_j + \epsilon_3 x_k; t_1, 1)) \\ \times \prod_{\substack{v_1, \dots, v_4 = \pm 1 \\ v_i = \epsilon_1, v_j = \epsilon_2}} g\left(\frac{1}{2} \sum_{k=1}^4 v_k x_k; t_3, 1\right) g(\epsilon_1 x_i + \epsilon_2 x_j; t_1, 1) \\ \times g(\epsilon_1 x_i + \epsilon_2 x_j; q^{-1}t_1, q^{-1})(\mathcal{T}_{x_i}^{-\epsilon_1 \hbar} \mathcal{T}_{x_j}^{-\epsilon_2 \hbar} - 1) + m_{-\theta^\vee, t}, \quad (4.38)$$

where $g(z; a, b)$ is defined by (4.27).

Let us consider the simplest example $S_0 = \{\alpha_3\} \cong A_1$. Then $\mathcal{D}_0 = \{x \in V \mid x_4 = \eta_3\}$ and the condition (3.13) becomes $2\eta_3 = \hbar$. The restriction of the operator (4.38) takes the form

$$\begin{aligned} \overline{M_{-\theta^\vee}} = & \sum_{\substack{1 \leq i < j \leq 3 \\ \epsilon_1, \epsilon_2 = \pm 1}} g(\epsilon_1 x_i; q^{1/2}, 1) g(\epsilon_1 x_i; t_1 q^{1/2}, q^{1/2}) g(\epsilon_1 x_i; t_1 q^{-1/2}, q^{-1/2}) \\ & \times g(\epsilon_2 x_j; q^{1/2}, 1) g(\epsilon_2 x_j; t_1 q^{1/2}, q^{1/2}) g(\epsilon_2 x_j; t_1 q^{-1/2}, q^{-1/2}) \\ & \times g(\epsilon_1 x_i + \epsilon_2 x_j; t_1, 1) g(\epsilon_1 x_i + \epsilon_2 x_j; q^{-1} t_1, q^{-1}) \\ & \times \prod_{\epsilon_3 = \pm 1} \left(g(\epsilon_1 x_i + \epsilon_3 x_k; t_1, 1) g(\epsilon_2 x_j + \epsilon_3 x_k; t_1, 1) \right. \\ & \times g\left(\frac{1}{2}(\epsilon_1 x_i + \epsilon_2 x_j + \epsilon_3 x_k); q^{3/4}, q^{-1/4}\right) \left. \right) (\mathcal{T}_{x_i}^{-\epsilon_1 \hbar} \mathcal{T}_{x_j}^{-\epsilon_2 \hbar} - 1) \\ & + (q^{1/2} + q^{-1/2}) \sum_{\substack{1 \leq i \leq 3 \\ \epsilon_1 = \pm 1}} g(\epsilon_1 x_i; q^{1/2}, 1) \\ & \times \prod_{\substack{l \neq i, 4 \\ \epsilon_3 = \pm 1}} (g(\epsilon_1 x_i + \epsilon_3 x_l; t_1, 1) g(\epsilon_3 x_l; q^{1/2} t_1, q^{1/2})) \\ & \times \prod_{\substack{v_1, \dots, v_3 = \pm 1 \\ v_i = \epsilon_1}} g\left(\frac{1}{2} \sum_{l=1}^3 v_l x_l; q^{3/4}, q^{1/4}\right) g(\epsilon_1 x_i; t_1 q^{1/2}, q^{1/2}) \\ & \times g(\epsilon_1 x_i; q^{-1/2} t_1, q^{-1/2}) (\mathcal{T}_{x_i}^{-\epsilon_1 \hbar} - 1) + m_{-\theta^\vee, t}, \end{aligned} \quad (4.39)$$

where in every term in the sum k is such that $1 \leq k \leq 3$, $k \neq i, j$; and $t_1 = e^{\eta_1/2}$, $q^{1/2} = e^{\hbar/4}$ are parameters.

Remark 4.3. It is possible to obtain a commuting operator to (4.39) by restriction of the relevant Macdonald operators for the small weights [32]. Similarly one can get Hamiltonians and quantum integrals starting from the Macdonald operators for the small weights in the case of other exceptional root systems.

5. The case (C_n^\vee, C_n)

The affine root system of type (C_n^\vee, C_n) is given by

$$\begin{aligned} \mathbb{S} = & \{[\epsilon_1 e_i + \epsilon_2 e_j, \hbar k] \mid 1 \leq i < j \leq n; \epsilon_1, \epsilon_2 = \pm 1; k \in \mathbb{Z}\} \\ & \cup \left\{ \epsilon \cdot \left[e_i, \frac{\hbar k}{2} \right] \mid 1 \leq i \leq n; \epsilon = \pm 1, \pm 2; k \in \mathbb{Z} \right\} \subset \widehat{V} \end{aligned} \quad (5.1)$$

(see [23, 25]). Let \widetilde{W} be the group generated by the affine reflections s_α with $\alpha \in \mathbb{S}$. The corresponding DAHA is defined in [28] (see also [26]). We recall the associated commuting difference operators below.

Let $R = C_n$, that is

$$R = \{\epsilon_1 e_i + \epsilon_2 e_j \mid 1 \leq i < j \leq n; \epsilon_1, \epsilon_2 = \pm 1\} \cup \{\epsilon 2e_i \mid 1 \leq i \leq n; \epsilon = \pm 1\}, \quad (5.2)$$

$$\Delta = \{\alpha_i = e_i - e_{i+1} \mid i = 1, \dots, n-1\} \cup \{\alpha_n = 2e_n\} \subset R, \quad (5.3)$$

let $\widehat{R} = \{[\alpha, \hbar k] \mid \alpha \in R; k \in \mathbb{Z}\}$ and $\widetilde{\Delta} = \Delta \cup \{\alpha_0 = [-\theta, \hbar] = [-2e_1, \hbar]\}$. Then $\widetilde{W} = \{s_\alpha \mid \alpha \in \mathbb{S}\} = \{s_\alpha \mid \alpha \in \widehat{R}\}$ is the affine Weyl group for C_n .

Let Q^\vee, P^\vee, P and W be the lattices of coroots, of coweights, of weights and the Weyl group for the root system $R = C_n$ respectively. Let also $\widehat{P} \subset \widehat{V}$ be the subset defined in Section 2.1 for $R = C_n$.

5.1. Noumi representation of (C_n^\vee, C_n) type DAHA and Koornwinder operator

Let $t, u : \widehat{R} \rightarrow \mathbb{C} \setminus \{0\}$ be such that $t_\alpha = t_{\tilde{w}\alpha} = t(\alpha)$, $u_\alpha = u_{\tilde{w}\alpha} = u(\alpha)$ for any $\tilde{w} \in \widetilde{W}$, $\alpha \in \widehat{R}$. Put $u_{\pm e_i \pm e_j} = 1$, $t_i = t_{\alpha_i}$, $u_i = u_{\alpha_i}$. So we have $t_1 = t_2 = \dots = t_{n-1}$ and $u_1 = u_2 = \dots = u_{n-1} = 1$.

Definition 5.1. (See [28].) Double Affine Hecke Algebra of type (C_n^\vee, C_n) is the algebra $\mathcal{H}_{t,u}$ generated by the elements T_0, T_1, \dots, T_n , and by the set of pairwise commuting elements $\{X^\mu\}_{\mu \in \widehat{P}}$ with the relations (2.4)–(2.7), (2.10) and

$$T_i^{-1} X^\mu T_i^{-1} = X^{\mu - \alpha_i} + (u_i - u_i^{-1}) X^{\mu - \alpha_i/2} T_i^{-1}, \quad \text{if } (\mu, \alpha_i^\vee) = 1, \mu \in \widehat{P}, \quad (5.4)$$

where $i = 0, 1, \dots, n$.

The algebra $\mathcal{H}_{t,u}$ depends on six independent parameters t_1, t_n, t_0, u_n, u_0 and \hbar . Note that the elements π_r are not included in this case. One can also define an ‘extended’ version of (C_n^\vee, C_n) type DAHA where affine Weyl group $\widetilde{W} = W \ltimes \tau(Q^\vee)$ is replaced with the extended affine Weyl group $\widehat{W} = W \ltimes \tau(P^\vee)$. However the \widehat{W} -invariance of the functions t, u leads to less independent parameters in this algebra and subsequently in the commuting difference operators. Note also that if $u_0 = u_n = 1$ then the algebra $\mathcal{H}_{t,u}$ can be interpreted as a ‘non-extended’ version of DAHA of type \widehat{C}_n .

Consider the action of the generators of $\mathcal{H}_{t,u}$ on the space \mathcal{F} given by the Noumi representation [26]

$$X^\mu = e^{\mu(x)}, \quad T_i = t_i + t_i^{-1} \frac{(1 - t_i u_i e^{-\alpha_i(x)/2})(1 + t_i u_i^{-1} e^{-\alpha_i(x)/2})}{1 - e^{-\alpha_i(x)}} (s_i - 1), \quad (5.5)$$

where $\mu \in \widehat{P}$, $i = 0, 1, \dots, n$.

The elements $T_{\tilde{w}} \in \mathcal{H}_{t,u}$ with $\tilde{w} \in \widetilde{W}$ and Cherednik–Dunkl operators $Y^\lambda \in \mathcal{H}_{t,u}$ with $\lambda \in Q^\vee$ are defined by the formula (2.12), where $\pi_r = 1$, and

$$Y^{\lambda - \nu} = T_{\tau(\lambda)} T_{\tau(\nu)}^{-1}, \quad \lambda, \nu \in P_+^\vee \cap Q^\vee. \quad (5.6)$$

The operators Y^λ commute with each other and satisfy the relations [26,25]

$$T_i Y^\lambda - Y^{s_i \lambda} T_i = \frac{t_i - t_i^{-1} + (\bar{t}_i - \bar{t}_i^{-1}) Y^{-\alpha_i^\vee}}{1 - Y^{-2\alpha_i^\vee}} (Y^\lambda - Y^{s_i \lambda}), \quad i = 1, \dots, n, \quad (5.7)$$

where $\lambda \in Q^\vee$, $\bar{t}_i = t_i$ for $i = 1, \dots, n-1$ and $\bar{t}_n = t_0$.

Let $\mathbb{C}[Y]$ be the commutative subalgebra of $\mathcal{H}_{t,u}$ spanned by the Cherednik–Dunkl operators (5.6). The relation (5.7) implies that the elements of the subalgebra $\mathbb{C}[Y]^W \subset \mathbb{C}[Y]$ preserve the subspace of W -invariant functions \mathcal{F} , that is $m_\lambda(Y) \mathcal{F}^W \subset \mathcal{F}^W$, where $\lambda \in Q^\vee$ and $m_\lambda(Y)$ is the operator defined by the formula (2.22). By restricting these operators to the subspace \mathcal{F}^W we obtain the difference operators M_λ given by formula (2.24) with (2.23).

Thus we get a commutative algebra \mathfrak{R} spanned by M_λ with $\lambda \in Q^\vee$. It is generated by the algebraically independent elements $M_{v_i} = M_{-v_i}$, where $v_i = e_1 + e_2 + \dots + e_i$ and $i = 1, \dots, n$. The first of the generators is the *Koornwinder operator* [26,21]

$$\begin{aligned} M_{-e_1} = M_{-\theta^\vee} &= \sum_{\substack{i=1 \\ \epsilon=\pm 1}}^n (t_0 t_n)^{-1} \prod_{\substack{j=1 \\ j \neq i}}^n (g(\epsilon x_i - x_j; t_1, 1) g(\epsilon x_i + x_j; t_1, 1)) \\ &\times \frac{(1 - t_0 u_0 q^{-1} e^{\epsilon x_i})(1 + t_0 u_0^{-1} q^{-1} e^{\epsilon x_i})(1 - t_n u_n e^{\epsilon x_i})(1 + t_n u_n^{-1} e^{\epsilon x_i})}{(1 - q^{-2} e^{2\epsilon x_i})(1 - e^{2\epsilon x_i})} \\ &\times (\mathcal{T}_{x_i}^{-\epsilon h} - 1) + \kappa_t, \end{aligned} \quad (5.8)$$

where $g(z; a, b) = \frac{ae^z - a^{-1}}{be^z - b^{-1}}$ and $\kappa_t = \frac{t_1^n - t_1^{-n}}{t_1 - t_1^{-1}}(t_0 t_n t_1^{n-1} + t_0^{-1} t_n^{-1} t_1^{1-n})$.

5.2. Invariant ideals and generalized Koornwinder operator

Let $\eta : R \rightarrow \mathbb{C}$ be a W -invariant function. It is defined by independent parameters $\eta_1 = \eta_{\alpha_1}$ and $\eta_n = \eta_{\alpha_n}$. Let $S_0 \subset \Delta$ and let \mathcal{D}_0 , W_+ , \mathcal{D} and $I_{\mathcal{D}}$ be defined in the same way as in Section 3 for the case $R = C_n$. Let $S_0 = \bigsqcup_{\ell=1}^m S_\ell$ and $R_0 = \bigsqcup_{\ell=1}^m R_\ell$ be the decompositions defined in Section 3.2. We denote $k_\ell = |S_\ell|$.

There are two possible cases: $\alpha_n \notin S_0$ and $\alpha_n \in S_0$. In the former case the components $R_\ell \cong A_{k_\ell}$ for all $\ell = 1, \dots, m$. In the latter case one of the components is of type C . We suppose $\alpha_n \in S_m$ so that $R_\ell \cong A_{k_\ell}$ for all $\ell = 1, \dots, m-1$ and $R_m \cong C_{k_m}$.

Lemma 5.1. Assume that

$$h_\eta^\ell = \hbar, \quad \text{whenever } \alpha_n \notin R_\ell. \quad (5.9)$$

Let $w \in W_+$ be such that $w\theta_\ell \neq \theta$ for all $\ell = 1, \dots, m$. Then $s_0 H_\eta(wS_0) \subset \mathcal{D}$.

Proof. If $\alpha_n \notin S_0$ then the assumptions of Lemma 3.9 hold and, therefore, $s_0 H_\eta(wS_0) \subset \mathcal{D}$. If $\alpha_n \in S_0$ then the statement can be proved in the same way as Lemma 3.9. Indeed, in the proof of Lemma 3.9 the condition $h_\eta^\ell = \hbar$ was used only if $(w\theta_\ell, \theta^\vee) = 1$. Thus it is sufficient to establish that $(w\theta_m, \theta^\vee) = 0$. Since wS_m consists of positive roots only this set has the form

$$wS_m = \{e_{i_1} - e_{i_2}, e_{i_2} - e_{i_3}, \dots, e_{i_{k_m-1}} - e_{i_{k_m}}, 2e_{i_{k_m}}\}, \quad (5.10)$$

for some $1 \leq i_1 < i_2 < \dots < i_{k_m} \leq n$. Since $w\theta_m = 2e_{i_1}$ the condition $w\theta_m \neq \theta = 2e_1$ implies $i_1 \neq 1$ and, hence, $(w\theta_m, \theta^\vee) = (2e_{i_1}, e_1) = 0$. \square

Theorem 5.2. Suppose the subset $S_0 \subset \Delta$ and the function η are such that the condition (5.9) is satisfied. Suppose the parameters of DAHA $\mathcal{H}_{t,u}$ are such that $t_1^2 = e^{\eta_1}$ if $n > 1$ and $\epsilon_n t_n u_n^{\epsilon_n} = e^{\eta_n/2}$ for some $\epsilon_n = \pm 1$. Suppose also that if $\alpha_n \in S_0$ then

$$\epsilon_0 \epsilon_n t_0 u_n^{\epsilon_n} u_0^{\epsilon_0} t_1^{2(\tilde{k}-1)} q^{-1} = 1 \quad (5.11)$$

for some $\epsilon_0 = \pm 1$, where $\tilde{k} = |S_m|$ is the rank of the irreducible component of type C . Then the ideal $I_{\mathcal{D}} \subset \mathcal{F}$ is invariant under the action of $\mathcal{H}_{t,u}$ in the Noumi representation (5.5).

Proof. Let $f \in I_{\mathcal{D}}$ and $x \in w\mathcal{D}_0$ for some $w \in W_+$. The operators T_1, \dots, T_{n-1} have the same form as the operators T_1, \dots, T_{n-1} of Section 3 in the case $R = C_n$, and the proof of invariance under their action follows by Proposition 3.5.

To prove the invariance under T_n note that if $\alpha_n \notin wS_0$ then we have $f(x) = f(s_n x) = 0$ and, hence, $T_n f(x) = 0$ (see the proof of Proposition 3.5). If $\alpha_n \in wS_0$ then $1 - t_n u_n e^{-x_n} = 0$ or $1 + t_n u_n^{-1} e^{-x_n} = 0$ hence $T_n f(x) = 0$.

To prove the invariance under T_0 suppose first that $w\theta_\ell = \theta$ for some ℓ . This is possible only if $\alpha_n \in S_0$ and $\ell = m$ since $\alpha_n \in S_m$. In this case we obtain $-\alpha_0(x) = (w\theta_m, x) - \hbar = h_\eta^m - \eta_\theta - \hbar = 2(\tilde{k} - 1)\eta_1 + \eta_n - \hbar$. Then the condition (5.11) implies that $\epsilon_0 t_0 u_0^{\epsilon_0} e^{-\alpha_0(x)/2} = 1$ for some $\epsilon_0 = \pm 1$. Now $T_0 f(x) = 0$ since $t_0 u_0 e^{-\alpha_0(x)/2} = 1$ or $t_0 u_0^{-1} e^{-\alpha_0(x)/2} = -1$. In the case when $w\theta_\ell \neq \theta$ for all $\ell = 1, \dots, m$ the plane $s_0(w\mathcal{D}_0) = s_0 H_\eta(wS_0)$ is contained in \mathcal{D} due to Lemma 5.1 and hence $f(s_0 x) = 0$. As in the proof of Proposition 3.10 it follows that $T_0 f(x) = 0$ for all $x \in w\mathcal{D}_0$. \square

Remark 5.1. The condition (5.11) can be rewritten in the form $\epsilon_0 t_0 u_0^{\epsilon_0} = e^{\eta_0/2}$ where $\eta_0 \in \mathbb{C}$ is such that $2(\tilde{k} - 1)\eta_1 + \eta_n + \eta_0 = \hbar$. Thus the number $\hat{h}_\eta^m = 2(\tilde{k} - 1)\eta_1 + \eta_n + \eta_0$ can be interpreted as a generalized Coxeter number for the affine root system $\hat{R}_m \subset \mathbb{S}$ of type $(C_k^\vee, C_{\tilde{k}})$.

Remark 5.2. Submodules in the polynomial representation of DAHA of type (C_n^\vee, C_n) were investigated by Kasatani [19]. In the case $\alpha_n \notin S_0$ the invariant ideals $I_{\mathcal{D}}$ correspond to the Kasatani's submodules $I_m^{(k,r)}$ with $r = 2$ (see [19, Proposition 4.13]). In the case $\alpha_n \in S_0$, $m = 1$ the ideals $I_{\mathcal{D}}$ correspond to the submodules V with $r = 2$ constructed in the paper [19, Section 4.8], the conditions (5.11) correspond to the conditions (49), (50) of [19] with $r = 2$.

Theorem 5.2 guarantees that the arguments of Section 4.1 remain valid in the (C_n^\vee, C_n) case. Thus we obtain the commutative algebra $\overline{\mathfrak{R}}$ spanned by the operators \overline{M}_λ , $\lambda \in Q^\vee$, defined by the formula (4.7).

To establish the integrability we use the method of Section 4.2 where the lattice P^\vee is replaced by Q^\vee . Note that $v_i = e_1 + e_2 + \dots + e_i$, $i = 1, \dots, n$, form the basis of Q^\vee . As before we denote $J = \{j = 1, \dots, n \mid \alpha_j \notin S_0\} = \{j_1, \dots, j_{\tilde{n}}\}$, where $\tilde{n} = \dim \mathcal{D}_0$.

Theorem 5.3. The algebra $\overline{\mathfrak{R}}$ contains \tilde{n} algebraically independent elements, namely, the operators $\overline{M}_{-v_{j_1}}, \dots, \overline{M}_{-v_{j_{\tilde{n}}}}$.

Indeed, the operators M_λ in this case have the form (2.31) with

$$g^\lambda(x) = \prod_{\alpha \in \hat{R}_\lambda} \frac{(1 - t_\alpha u_\alpha e^{\alpha(x)/2})(1 + t_\alpha u_\alpha^{-1} e^{\alpha(x)/2})}{1 - e^{\alpha(x)}}.$$

If the function $g^{-v_j}(x)$ is identically zero on \mathcal{D}_0 for some $j \in J$ then the function $1 - t_\alpha u_\alpha e^{\alpha(x)/2}$ or $1 + t_\alpha u_\alpha^{-1} e^{\alpha(x)/2}$ is identically zero on \mathcal{D}_0 for some $\alpha = [\alpha^{(1)}, \alpha^{(0)}] \in \hat{R}_{-v_j}$. It is possible only if α is a linear combination of the elements of S_0 and hence $(\alpha^{(1)}, v_j) = 0$. From the definition of \hat{R}_λ it follows that $(\alpha^{(1)}, v_j) < 0$, hence $g^{-v_j}|_{\mathcal{D}_0} \neq 0$, and the operators M_{-v_j} , $j \in J$, are algebraically independent (see the proof of Theorem 4.3 for details).

The algebra $\overline{\mathfrak{R}}$ contains the generalized Koornwinder operator. Let us introduce this operator explicitly. It acts on functions of two sets of variables x_1, \dots, x_{N_1} , y_1, \dots, y_{N_2} and it depends on

six independent parameters: two shift parameters \hbar, ξ and the parameters a, b, c, d . It is defined as

$$\mathcal{M}_{N_1, N_2} = \sum_{\substack{i=1 \\ \epsilon=\pm 1}}^{N_1} A_i^\epsilon(x, y) (\mathcal{T}_{x_i}^{-\epsilon\hbar} - 1) + \frac{1-q^{-2}}{1-s^{-2}} \sum_{\substack{\ell=1 \\ \epsilon=\pm 1}}^{N_2} B_\ell^\epsilon(x, y) (\mathcal{T}_{y_\ell}^{-\epsilon\xi} - 1), \quad (5.12)$$

where $q = e^{\hbar/2}$, $s = e^{\xi/2}$ and

$$\begin{aligned} A_i^\epsilon(x, y) &= \prod_{\substack{j=1 \\ j \neq i}}^{N_1} g(\epsilon x_i - x_j; s, 1) g(\epsilon x_i + x_j; s, 1) \\ &\times \prod_{\ell'=1}^{N_2} g(\epsilon x_i - y_{\ell'}; q^{1/2} s^{1/2}, q^{-1/2} s^{1/2}) g(\epsilon x_i + y_{\ell'}; q^{1/2} s^{1/2}, q^{-1/2} s^{1/2}) \\ &\times \frac{(1 - a e^{\epsilon x_i})(1 - b e^{\epsilon x_i})(1 - c e^{\epsilon x_i})(1 - d e^{\epsilon x_i})}{(1 - q^{-2} e^{2\epsilon x_i})(1 - e^{2\epsilon x_i})}, \end{aligned} \quad (5.13)$$

$$\begin{aligned} B_\ell^\epsilon(x, y) &= \prod_{j=1}^{N_1} g(\epsilon y_\ell - x_j; q^{1/2} s^{1/2}, q^{1/2} s^{-1/2}) g(\epsilon y_\ell + x_j; q^{1/2} s^{1/2}, q^{1/2} s^{-1/2}) \\ &\times \prod_{\substack{\ell'=1 \\ \ell' \neq \ell}}^{N_2} g(\epsilon y_\ell - y_{\ell'}; q, 1) g(\epsilon y_\ell + y_{\ell'}; q, 1) \\ &\times \frac{(1 - a q s^{-1} e^{\epsilon y_\ell})(1 - b q s^{-1} e^{\epsilon y_\ell})(1 - c q s^{-1} e^{\epsilon y_\ell})(1 - d q s^{-1} e^{\epsilon y_\ell})}{(1 - s^{-2} e^{2\epsilon y_\ell})(1 - e^{2\epsilon y_\ell})}. \end{aligned} \quad (5.14)$$

For $N_2 = 0$ this operator coincides with the Koornwinder operator (5.8) up to a constant and a factor (where a, b, c, d, ξ are expressed through the parameters of DAHA $\mathcal{H}_{t,u}$).

We note that this operator is invariant (up to a factor) under the involution

$$\begin{aligned} N_1 &\leftrightarrow N_2, & x &\leftrightarrow y, & \xi &\leftrightarrow \hbar, & s &\leftrightarrow q, & a &\mapsto a q s^{-1}, \\ b &\mapsto b q s^{-1}, & c &\mapsto c q s^{-1}, & d &\mapsto d q s^{-1}. \end{aligned} \quad (5.15)$$

Now we explain how this operator arises by the restriction of the Koornwinder operator M_{-e_1} . Consider first the case $\alpha_n \notin S_0$. In this case we assume that all the subsets S_ℓ coincide with (4.22) and $t_1 = e^{\eta_1/2}$. Then $N_1 = n - mk$, $N_2 = m$, $\hbar = k\eta_1$, $q = t_1^k$. We introduce orthogonal coordinates x_1, \dots, x_{N_1} , y_1, \dots, y_{N_2} on the corresponding plane \mathcal{D}_0 by (4.23).

Proposition 5.4. *In the case $\alpha_n \notin S_0$ we have $\overline{M_{-e_1}} = (t_0 t_n)^{-1} \mathcal{M}_{N_1, N_2} + \kappa_t$, where \mathcal{M}_{N_1, N_2} is the generalized Koornwinder operator (5.12) with $N_1 = n - mk$, $N_2 = m$ and the parameters*

$$\begin{aligned} s &= t_1, & \xi &= \eta_1, & a &= t_0 u_0 q^{-1}, \\ b &= -t_0 u_0^{-1} q^{-1}, & c &= t_n u_n, & d &= -t_n u_n^{-1} \end{aligned}$$

satisfying the condition $\hbar/\eta_1 = k \in \mathbb{Z}_{\geq 2}$.

In the case $\alpha_n \in S_0$ the subsets S_ℓ with $\ell = 1, \dots, m-1$ coincide with (4.22) and

$$S_m = \{e_{n-\tilde{k}+1} - e_{n-\tilde{k}+2}, e_{n-\tilde{k}+2} - e_{n-\tilde{k}+3}, \dots, e_{n-1} - e_n, 2e_n\}, \quad (5.16)$$

where $\tilde{k} = k_m = |S_m|$; $t_1 = e^{\eta_1/2}$, $\epsilon_n t_n u_n^{\epsilon_n} = e^{\eta_n/2}$. Then

$$N_1 = n - (m-1)k - \tilde{k}, \quad N_2 = m-1, \quad (5.17)$$

and the orthogonal coordinates $x_1, \dots, x_{N_1}, y_1, \dots, y_{N_2}$ on \mathcal{D}_0 are given by (4.23).

Proposition 5.5. *In the case $\alpha_n \in S_0$ we have $\overline{M_{-e_1}} = (t_0 t_n t_1^{2\tilde{k}})^{-1} \mathcal{M}_{N_1, N_2} + \kappa_t$, where \mathcal{M}_{N_1, N_2} is the generalized Koornwinder operator (5.12) with N_1, N_2 given by (5.17) and the parameters*

$$\begin{aligned} s &= t_1, & \xi &= \eta_1, & a &= \epsilon_n t_1^2 t_n^{-1} u_n^{-\epsilon_n}, \\ b &= -\epsilon_n t_n u_n^{-\epsilon_n}, & c &= \epsilon_0 q t_1^2 t_0^{-1} u_0^{-\epsilon_0}, & d &= -q^{-1} \epsilon_0 t_0 u_0^{-\epsilon_0} \end{aligned}$$

where the condition (5.11) holds for $\tilde{k} \in \mathbb{Z}_{\geq 1}$, and if $m > 1$ then $\hbar/\eta_1 = k \in \mathbb{Z}_{\geq 2}$.

The propositions can be established by direct calculations. In both cases the obtained operators $\overline{M_{-e_1}}$ depend on six independent parameters. Namely, in the case $\alpha_n \notin S_0$ we have five continuous parameters ξ, a, b, c, d and one discrete parameter k (where $\hbar = k\xi$), and in the case $\alpha_n \in S_0$ we have four continuous parameters ξ, b, c, d and two discrete ones k, \tilde{k} ($\hbar = k\xi$, $a = c^{-1}e^{(\tilde{k}+1)\xi}$).

Remark 5.3. The operator (5.12) generalizes the deformed rational Koornwinder operator [30, formula (94)]. Namely the latter can be obtained by the substitution $x_i \rightarrow x_i/\omega$, $y_i \rightarrow y_i/\omega$, $\xi \rightarrow \xi/\omega$, $\hbar \rightarrow \hbar/\omega$, $a = e^{a'/\omega}$, $b = -e^{b'/\omega}$, $c = e^{c'/\omega}$, $d = -e^{d'/\omega}$ to (5.12) and by taking the limit $\omega \rightarrow \infty$.

It follows from above that the operator \mathcal{M}_{N_1, N_2} comes together with the complete algebra of commuting difference operators. We specify these operators explicitly and establish commutativity (without extra integrality restriction for a combination of parameters). We start with the explicit formula for the higher Koornwinder operators obtained by van Diejen [31]:

$$H_r = \sum_{\substack{J_1 \subset \{1, \dots, n\}, \\ \epsilon_i = \pm 1, i \in J_1}} \prod_{|J_1|=r} G(e^{\epsilon_i x_i}) \sum_{J_2 \subset J_1} (-1)^{r-|J_2|} \mathcal{U}_1 \mathcal{U}_2 \prod_{i \in J_2} \mathcal{T}_{x_i}^{-\epsilon_i \hbar}, \quad (5.18)$$

where $r = 1, \dots, n$,

$$G(z) = \frac{(1 - t_0 u_0 q^{-1} z)(1 + t_0 u_0^{-1} q^{-1} z)(1 - t_n u_n z)(1 + t_n u_n^{-1} z)}{(1 - q^{-2} z^2)(1 - z^2)}, \quad (5.19)$$

$$\begin{aligned} \mathcal{U}_b &= \prod_{\substack{i, j \in J_b \setminus J_{b+1} \\ i < j}} g(\epsilon_i x_i + \epsilon_j x_j; t_1, 1) g(\epsilon_i x_i + \epsilon_j x_j; q^{-1} t_1^{2b-3}, q^{-1}) \\ &\times \prod_{\substack{i \in J_b \setminus J_{b+1} \\ j \notin J_b}} g(\epsilon_i x_i + x_j; t_1, 1) g(\epsilon_i x_i - x_j; t_1, 1), \end{aligned} \quad (5.20)$$

where $b = 1, 2$, $J_3 = \emptyset$ and $q = e^{\hbar/2}$. These operators commute with each other, and for $r = 1$ we have $H_1 = t_0 t_n (M_{-e_1} - \kappa_t)$, where M_{-e_1} is the Koornwinder operator (5.8).

Note that the operators have the form $H_r = g_{-v_r}^r(x)\tau(-v_r) + \sum_{\lambda > -v_r} g_{\lambda}^r(x)\tau(\lambda)$, where $v_r = e_1 + \dots + e_r$ and $g_{\lambda}^r(x)$ are meromorphic functions such that $g_{-v_r}^r(x) \not\equiv 0$. Hence the operators (5.18) with $r = 1, \dots, n$ are algebraically independent (cf. Section 2.2).

The operators H_r are W -invariant. Since for generic parameters the centralizer of M_{-e_1} in the algebra of W -invariant difference operators (with suitable coefficients) coincides with the algebra \mathfrak{R} (see [22, Theorem 3.15]) we obtain $H_r \in \mathfrak{R}$. Hence $H_r \in \mathfrak{R}$ for any values of the parameters, and the restrictions of the operators H_r onto the corresponding plane \mathcal{D}_0 can be defined and they commute with each other.

Consider restriction of the operators (5.18) in the case $\alpha_n \notin S_0$. The structure of the functions (5.20) implies that if \mathcal{U}_b is not zero on the plane \mathcal{D}_0 then J_b have the form

$$J_b = I_b \cup \bigcup_{\ell=1}^{N_2} \{N_1 + k\ell - s \mid 0 \leq s \leq m_{\ell b}^- - 1 \text{ or } k - m_{\ell b}^+ \leq s \leq k - 1\}, \quad b = 1, 2, \quad (5.21)$$

where $I_2 \subset I_1 \subset \{1, \dots, N_1\}$ and $m_{\ell b}^{\pm}$ are non-negative integers such that $m_{\ell 2}^{\pm} \leq m_{\ell 1}^{\pm}$, $m_{\ell 1}^- + m_{\ell 1}^+ \leq k$ and either $m_{\ell 2}^-$ or $m_{\ell 2}^+$ vanishes for each ℓ . Moreover, the signs ϵ_i are given by

$$\epsilon_{N_1+k\ell-s} = \begin{cases} -1, & \text{if } 0 \leq s \leq m_{\ell 1}^- - 1, \\ +1, & \text{if } k - m_{\ell 1}^+ \leq s \leq k - 1. \end{cases} \quad (5.22)$$

We recall the coordinates $x_1, \dots, x_{N_1}, y_1, \dots, y_{N_2}$ defined by (4.23) on the plane \mathcal{D}_0 .

Proposition 5.6. *The restriction of the operator H_r takes the form*

$$\begin{aligned} \overline{H_r} = & \sum_{\substack{I_1 \subset \{1, \dots, N_1\} \\ \epsilon_i = \pm 1, i \in I_1 \\ m_{\ell 1}^{\pm} \in \mathbb{Z}_{\geq 0}, 1 \leq \ell \leq N_2 \\ |I_1| + \sum_{\ell=1}^{N_2} (m_{\ell 1}^+ + m_{\ell 1}^-) = r}} \prod_{i \in I_1} G(e^{\epsilon_i x_i}) \prod_{\substack{\ell=1 \\ \epsilon = \pm 1}}^{N_2} \prod_{p=0}^{m_{\ell 1}^{\epsilon} - 1} G(q t_1^{-1-2p} e^{\epsilon y_{\ell}}) \\ & \times \sum_{\substack{I_2 \subset I_1, m_{\ell 2}^{\pm} \in \mathbb{Z}_{\geq 0} \\ m_{\ell 2}^+ \leq m_{\ell 1}^+, m_{\ell 2}^+ m_{\ell 2}^- = 0}} (-1)^{r - |I_2| - \sum_{\ell=1}^{N_2} (m_{\ell 2}^+ + m_{\ell 2}^-)} \overline{\mathcal{U}}_1 \overline{\mathcal{U}}_2 \\ & \times \prod_{i \in I_2} \mathcal{T}_{x_i}^{-\epsilon_i \hbar} \prod_{\ell=1}^{N_2} \mathcal{T}_{y_{\ell}}^{-(m_{\ell 2}^+ - m_{\ell 2}^-) \eta_1}, \end{aligned} \quad (5.23)$$

where

$$\begin{aligned} \overline{\mathcal{U}}_b = & \prod_{\ell=1}^{N_2} \prod_{p=m_{\ell, b+1}^+}^{m_{\ell b}^+ - 1} g(0; q t_1^{-p - m_{\ell, b+1}^-}, t_1^{-p + m_{\ell b}^+}) g(0; t_1^{-p-1 - m_{\ell b}^-}, t_1^{-p-1 - m_{\ell, b+1}^-}) \\ & \times \prod_{\ell=1}^{N_2} \prod_{p=m_{\ell, b+1}^-}^{m_{\ell b}^- - 1} g(0; q t_1^{-p - m_{\ell b}^+}, t_1^{-p + m_{\ell b}^-}) \end{aligned}$$

$$\begin{aligned}
& \times \prod_{\substack{i,j \in I_b \setminus I_{b+1} \\ i < j}} g(\epsilon_i x_i + \epsilon_j x_j; t_1, 1) g(\epsilon_i x_i + \epsilon_j x_j; q^{-1} t_1^{2b-3}, q^{-1}) \\
& \times \prod_{\substack{i \in I_b \setminus I_{b+1} \\ \epsilon = \pm 1 \\ j \notin I_b}} g(\epsilon_i x_i + \epsilon x_j; t_1, 1) \prod_{\substack{\ell=1 \\ \epsilon, \epsilon' = \pm 1 \\ j \notin I_b}}^{N_2} g(\epsilon y_\ell + \epsilon' x_j; q^{\frac{1}{2}} t_1^{\frac{1}{2} - m_{\ell, b+1}^\epsilon}, q^{\frac{1}{2}} t_1^{\frac{1}{2} - m_{\ell b}^\epsilon}) \\
& \times \prod_{\substack{\ell=1 \\ \epsilon = \pm 1 \\ i \in I_b \setminus I_{b+1}}}^{N_2} g(\epsilon_i x_i + \epsilon y_\ell; q^{\frac{1}{2}} t_1^{\frac{1}{2} - m_{\ell, b+1}^\epsilon}, q^{-\frac{1}{2}} t_1^{\frac{1}{2} + m_{\ell b}^\epsilon}) \\
& \times g(\epsilon_i x_i + \epsilon y_\ell; q^{-\frac{1}{2}} t_1^{\frac{2b-3}{2} - (2-b)m_{\ell 1}^\epsilon}, q^{-\frac{1}{2}} t_1^{\frac{2b-3}{2} - m_{\ell 2}^\epsilon}) \\
& \times \prod_{\substack{1 \leq \ell < \ell' \leq N_2 \\ \epsilon, \epsilon' = \pm 1}} \prod_{p=m_{\ell, b+1}^\epsilon}^{m_{\ell b}^\epsilon - 1} (g(\epsilon y_\ell + \epsilon' y_{\ell'}; q t_1^{-p - m_{\ell', b+1}^{\epsilon'}}, t_1^{-p + m_{\ell' b}^{\epsilon'}}) \\
& \times g(\epsilon y_\ell + \epsilon' y_{\ell'}; t_1^{-p - (2-b)(1 + m_{\ell' 1}^{\epsilon'})}, t_1^{-p - (2-b) - m_{\ell' 2}^{\epsilon'}})) \\
& \times \prod_{\substack{1 \leq \ell' < \ell \leq N_2 \\ \epsilon, \epsilon' = \pm 1}} \prod_{p=m_{\ell, b+1}^\epsilon}^{m_{\ell b}^\epsilon - 1} g(\epsilon y_\ell + \epsilon' y_{\ell'}; q t_1^{-p - m_{\ell' b}^{\epsilon'}}, t_1^{-p + m_{\ell' b}^{\epsilon'}}) \\
& \times \prod_{\substack{\ell=1 \\ \epsilon = \pm 1}}^{N_2} \prod_{p=m_{\ell, b+1}^\epsilon}^{m_{\ell b}^\epsilon - 1} g(2\epsilon y_\ell; q t_1^{-1-2p}, t_1^{-p + m_{\ell b}^\epsilon}) \\
& \times g(2\epsilon y_\ell; t_1^{-1-bp - (2-b)m_{\ell 1}^\epsilon}, t_1^{-2(2-b) - (3-b)p - (b-1)m_{\ell 2}^\epsilon}), \tag{5.24}
\end{aligned}$$

with $b = 1, 2$, $I_3 = \emptyset$, $m_{\ell 3}^\pm = 0$ and $q = e^{h/2}$, $t_1 = e^{\eta_1/2}$.

Note that we do not impose the condition $m_{\ell 1}^- + m_{\ell 1}^+ \leq k$ in the sum in (5.23) since the product $\bar{\mathcal{U}}_1 \bar{\mathcal{U}}_2$ vanishes if $m_{\ell 1}^- + m_{\ell 1}^+ > k$, so the operators \bar{H}_r do not have explicit dependence on k .

Proof of Proposition 5.6. The structure of the operator (5.23) follows from the formulae (5.21), (5.22). Let $i = N_1 + k\ell - s \in J_{\ell b}$ for some $\ell = 1, \dots, N_2$, $s = 0, \dots, k-1$, $b = 1, 2$. Then the formulae (5.21), (5.22) imply that on \mathcal{D}_0 one has $\epsilon x_i = \epsilon y_\ell + (\frac{k-1}{2} - p)\eta_1$ for some $p = 0, \dots, m_{\ell b}^\epsilon - 1$ with $\epsilon = \epsilon_i$. Hence the restriction of $\prod_{i \in J_1} G(e^{\epsilon_i x_i})$ takes the form

$$\prod_{i \in I_1} G(e^{\epsilon_i x_i}) \prod_{\substack{\ell=1 \\ \epsilon = \pm 1}}^{N_2} \prod_{p=0}^{m_{\ell 1}^\epsilon - 1} G(q t_1^{-1-2p} e^{\epsilon y_\ell}).$$

Consider the restriction of the first product in the expression (5.20) which is taken over $i, j \in J_b \setminus J_{b+1}$. Its part corresponding to the case $i < j \leq N_1$ is equal to

$$\prod_{\substack{i, j \in I_b \setminus I_{b+1} \\ i < j}} g(\epsilon_i x_i + \epsilon_j x_j; t_1, 1) g(\epsilon_i x_i + \epsilon_j x_j; q^{-1} t_1^{2b-3}, q^{-1}).$$

For $i \leq N_1 < j$ we obtain

$$\begin{aligned} & \prod_{i \in I_b \setminus I_{b+1}} \prod_{\substack{\ell=1 \\ \epsilon=\pm 1}}^{N_2} \prod_{p=m_{\ell, b+1}^{\epsilon}}^{m_{\ell b}^{\epsilon}-1} g(\epsilon_i x_i + \epsilon y_{\ell}; q^{\frac{1}{2}} t_1^{\frac{1}{2}-p}, q^{\frac{1}{2}} t_1^{-\frac{1}{2}-p}) \\ & \quad \times g(\epsilon_i x_i + \epsilon y_{\ell}; q^{-\frac{1}{2}} t_1^{-\frac{1}{2}-p+2b-3}, q^{-\frac{1}{2}} t_1^{-\frac{1}{2}-p}) \\ & = \prod_{\substack{\ell=1 \\ \epsilon=\pm 1 \\ i \in I_b \setminus I_{b+1}}}^{N_2} g(\epsilon_i x_i + \epsilon y_{\ell}; q^{\frac{1}{2}} t_1^{\frac{1}{2}-m_{\ell, b+1}^{\epsilon}}, q^{\frac{1}{2}} t_1^{\frac{1}{2}-m_{\ell b}^{\epsilon}}) \\ & \quad \times g(\epsilon_i x_i + \epsilon y_{\ell}; q^{-\frac{1}{2}} t_1^{\frac{2b-3}{2}-(2-b)m_{\ell 1}^{\epsilon}}, q^{-\frac{1}{2}} t_1^{\frac{2b-3}{2}-m_{\ell 2}^{\epsilon}}). \end{aligned} \quad (5.25)$$

Here we used the formulae

$$\begin{aligned} & \prod_{p=m}^M g(z; at_1^{1-p}, at_1^{-p}) = g(z; at_1^{1-M}, at_1^{-M}), \\ & \prod_{p=m}^M g(z; at_1^{-1-p}, at_1^{-p}) = g(z; at_1^{-1-M}, at_1^{-M}), \end{aligned}$$

where $m, M \in \mathbb{Z}$ are such that $m \leq M+1$, $a, z \in \mathbb{C}$ (the second formula is used in the case $b=1$ only). If $N_1 < i < j$ then $i = N_1 + k\ell - s$ and $j = N_1 + k\ell' - s'$, where $1 \leq \ell, \ell' \leq N_2$, $0 \leq s, s' \leq k-1$. If $\ell \neq \ell'$ then we obtain

$$\begin{aligned} & \prod_{\substack{1 \leq \ell < \ell' \leq N_2 \\ \epsilon, \epsilon' = \pm 1}} \prod_{p=m_{\ell, b+1}^{\epsilon}}^{m_{\ell b}^{\epsilon}-1} \prod_{p'=m_{\ell', b+1}^{\epsilon'}}^{m_{\ell' b}^{\epsilon'}-1} g(\epsilon y_{\ell} + \epsilon' y_{\ell'}; q t_1^{-p-p'}, q t_1^{-1-p-p'}) \\ & \quad \times g(\epsilon y_{\ell} + \epsilon' y_{\ell'}; t_1^{-1-p-p'+2b-3}, t_1^{-1-p-p'}) \\ & = \prod_{\substack{1 \leq \ell < \ell' \leq N_2 \\ \epsilon, \epsilon' = \pm 1}} \prod_{p=m_{\ell, b+1}^{\epsilon}}^{m_{\ell b}^{\epsilon}-1} (g(\epsilon y_{\ell} + \epsilon' y_{\ell'}; q t_1^{-p-m_{\ell', b+1}^{\epsilon'}}, q t_1^{-p-m_{\ell' b}^{\epsilon'}}) \\ & \quad \times g(\epsilon y_{\ell} + \epsilon' y_{\ell'}; t_1^{-p-(2-b)(1+m_{\ell' 1}^{\epsilon'})}, t_1^{-p-(2-b)-m_{\ell' 2}^{\epsilon'}})), \end{aligned} \quad (5.26)$$

while if $\ell = \ell'$ we derive the following two factors. The factor corresponding to $\epsilon_i = -\epsilon_j = +1$ is equal to

$$\begin{aligned} & \prod_{\ell=1}^{N_2} \prod_{p=m_{\ell,b+1}^+}^{m_{\ell b}^+-1} \prod_{p'=m_{\ell,b+1}^-}^{m_{\ell b}^--1} g(0; qt_1^{-p-p'}, qt_1^{-1-p-p'}) g(0; t_1^{-1-p-p'+2b-3}, t_1^{-1-p-p'}) \\ &= \prod_{\ell=1}^{N_2} \prod_{p=m_{\ell,b+1}^+}^{m_{\ell b}^+-1} g(0; qt_1^{-p-m_{\ell,b+1}^-}, qt_1^{-p-m_{\ell b}^-}) \\ & \quad \times g(0; t_1^{-p-1-m_{\ell b}^-}, t_1^{-p-1-m_{\ell,b+1}^-}). \end{aligned} \quad (5.27)$$

(In the case $b = 2$ both sides of the equality (5.27) equal 1.) The factor corresponding to $\epsilon_i = \epsilon_j = \epsilon = \pm 1$ is equal to

$$\begin{aligned} & \prod_{\substack{\ell=1 \\ \epsilon=\pm 1}}^{N_2} \prod_{p=m_{\ell,b+1}^\epsilon}^{m_{\ell b}^\epsilon-1} \prod_{p'=p+1}^{m_{\ell b}^\epsilon-1} g(2\epsilon y_\ell; qt_1^{-p-p'}, qt_1^{-1-p-p'}) g(2\epsilon y_\ell; t_1^{-1-p-p'+2b-3}, t_1^{-1-p-p'}) \\ &= \prod_{\substack{\ell=1 \\ \epsilon=\pm 1}}^{N_2} \prod_{p=m_{\ell,b+1}^\epsilon}^{m_{\ell b}^\epsilon-1} g(2\epsilon y_\ell; qt_1^{-1-2p}, qt_1^{-p-m_{\ell b}^\epsilon}) \\ & \quad \times g(2\epsilon y_\ell; t_1^{-1-bp-(2-b)m_{\ell 1}^\epsilon}, t_1^{-2(2-b)-(3-b)p-(b-1)m_{\ell 2}^\epsilon}). \end{aligned} \quad (5.28)$$

Now consider the restriction of the second product in the expression (5.20) which is taken over $i \in J_b \setminus J_{b+1}$, $j \notin J_b$. If $1 \leq i, j \leq N_1$ we obtain

$$\prod_{\substack{i \in I_b \setminus I_{b+1} \\ j \notin I_b}} g(\epsilon_i x_i + x_j; t_1, 1) g(\epsilon_i x_i - x_j; t_1, 1).$$

For $j \leq N_1 < i$ we obtain

$$\prod_{\substack{\ell=1 \\ \epsilon, \epsilon'=\pm 1}}^{N_2} \prod_{\substack{j=1 \\ j \notin I_b}}^{N_1} g(\epsilon y_\ell + \epsilon' x_j; q^{\frac{1}{2}} t_1^{\frac{1}{2}-m_{\ell,b+1}^\epsilon}, q^{\frac{1}{2}} t_1^{\frac{1}{2}-m_{\ell b}^\epsilon}).$$

If $i \leq N_1 < j$ then $\epsilon x_j = \epsilon y_\ell + (\frac{k-1}{2} - p)\eta_1$ with $p = m_{\ell b}^\epsilon, \dots, k-1 - m_{\ell b}^{-\epsilon}$ for any $\epsilon = \pm 1$ (it follows from $j \notin J_b$), so we obtain

$$\begin{aligned} & \prod_{\substack{\ell=1 \\ \epsilon=\pm 1 \\ i \in I_b \setminus I_{b+1}}}^{N_2} \prod_{p=m_{\ell b}^\epsilon}^{k-1-m_{\ell b}^{-\epsilon}} g(\epsilon_i x_i + \epsilon y_\ell; q^{\frac{1}{2}} t_1^{\frac{1}{2}-p}, q^{\frac{1}{2}} t_1^{-\frac{1}{2}-p}) \\ &= \prod_{\substack{\ell=1 \\ \epsilon=\pm 1 \\ i \in I_b \setminus I_{b+1}}}^{N_2} g(\epsilon_i x_i + \epsilon y_\ell; q^{\frac{1}{2}} t_1^{\frac{1}{2}-m_{\ell b}^\epsilon}, q^{-\frac{1}{2}} t_1^{\frac{1}{2}+m_{\ell b}^{-\epsilon}}). \end{aligned} \quad (5.29)$$

If $N_1 < i < j$ then $i = N_1 + k\ell - s$ and $j = N_1 + k\ell' - s'$, where $1 \leq \ell, \ell' \leq N_2, 0 \leq s, s' \leq k - 1$. In the case $\ell \neq \ell'$ we obtain

$$\begin{aligned} & \prod_{\substack{\ell, \ell'=1 \\ \ell \neq \ell' \\ \epsilon, \epsilon'=\pm 1}}^{N_2} \prod_{p=m_{\ell, b+1}^{\epsilon}}^{m_{\ell b}^{\epsilon}-1} \prod_{p'=m_{\ell' b}^{\epsilon'}}^{k-1-m_{\ell' b}^{-\epsilon'}} g(\epsilon y_{\ell} + \epsilon' y_{\ell'}; q t_1^{-p-p'}, q t_1^{-1-p-p'}) \\ &= \prod_{\substack{\ell, \ell'=1 \\ \ell \neq \ell' \\ \epsilon, \epsilon'=\pm 1}}^{N_2} \prod_{p=m_{\ell, b+1}^{\epsilon}}^{m_{\ell b}^{\epsilon}-1} g(\epsilon y_{\ell} + \epsilon' y_{\ell'}; q t_1^{-p-m_{\ell' b}^{\epsilon'}}, t_1^{-p+m_{\ell' b}^{-\epsilon'}}), \end{aligned} \quad (5.30)$$

and for $\ell = \ell'$ we derive

$$\begin{aligned} & \prod_{\substack{\ell=1 \\ \epsilon, \epsilon'=\pm 1}}^{N_2} \prod_{p=m_{\ell, b+1}^{\epsilon}}^{m_{\ell b}^{\epsilon}-1} \prod_{p'=m_{\ell b}^{\epsilon'}}^{k-1-m_{\ell b}^{-\epsilon'}} g(\epsilon y_{\ell} + \epsilon' y_{\ell}; q t_1^{-p-p'}, q t_1^{-1-p-p'}) \\ &= \prod_{\substack{\ell=1 \\ \epsilon=\pm 1}}^{N_2} \prod_{p=m_{\ell, b+1}^{\epsilon}}^{m_{\ell b}^{\epsilon}-1} g(0; q t_1^{-p-m_{\ell b}^{-\epsilon}}, t_1^{-p+m_{\ell b}^{\epsilon}}) \\ &\quad \times \prod_{\substack{\ell=1 \\ \epsilon=\pm 1}}^{N_2} \prod_{p=m_{\ell, b+1}^{\epsilon}}^{m_{\ell b}^{\epsilon}-1} g(2\epsilon y_{\ell}; q t_1^{-p-m_{\ell b}^{\epsilon}}, t_1^{-p+m_{\ell b}^{-\epsilon}}) \end{aligned} \quad (5.31)$$

(the first factor in the right-hand side corresponds to $\epsilon' = -\epsilon$ and the second one corresponds to $\epsilon' = \epsilon$). By combining the factors (5.27), (5.28) and (5.31) we obtain the beginning and ending of the formula (5.24). Then combining (5.26) and the factors of (5.30) with $\ell < \ell'$ we obtain the corresponding factors of (5.24). By combining (5.25) with (5.29) and taking into account all the remaining factors we obtain the formula (5.24). \square

For $r = 1$ we obtain the generalized Koornwinder operator $\overline{H}_1 = \mathcal{M}_{N_1, N_2}$ with the parameters as in Proposition 5.4, so we get a collection of operators commuting with the operator (5.12). Note that if $\hbar = \eta_1$ the operators \overline{H}_r coincide with the specialization of H_r for $n = N_1 + N_2$. Thus we obtain the following theorem.

Theorem 5.7. For any values of the parameters $\hbar, \eta_1, t_0, t_n, u_0, u_n$ ($u_0, u_n \neq 0$) the operators \overline{H}_r given by the formulae (5.23), (5.24), $r \in \mathbb{Z}_{>0}$, pairwise commute. The operators \overline{H}_r with $r = 1, \dots, N_1 + N_2$ are algebraically independent at generic values of the parameters.

6. The case of DAHA of \widehat{R}^{\vee} type

Let $\widehat{R}^{\vee} = \{\alpha^{\vee} \mid \alpha \in \widehat{R}\} \subset \widehat{V}$, where R is an irreducible reduced root system in $\mathbb{R}^n \subset V \cong \mathbb{C}^n$ with the set of simple roots $\Delta = \{\alpha_1, \dots, \alpha_n\}$. We have $\widehat{R}^{\vee} \not\cong \widehat{R}$ only for $R = B_n, C_n, F_4$ and G_2 , but we will consider the general R .

Let $W, \widehat{W}, \Pi, \mathcal{O}$ be the same as in Section 2 and let $t : \widehat{R}^\vee \rightarrow \mathbb{C} \setminus \{0\}$ be a \widehat{W} -invariant function. Put $t_i = t_{\alpha_i^\vee} = t(\alpha_i^\vee)$ and $\widehat{P}^\vee = \{\mu = [\mu^{(1)}, \mu^{(0)}] \mid \mu^{(1)} \in P^\vee, \mu^{(0)} \in \frac{\hbar}{\epsilon_0} \mathbb{Z}\} \subset \widehat{V}$, where $\epsilon_0 \in \mathbb{Z}$ is such that $(P^\vee, P^\vee) \subset \epsilon_0^{-1} \mathbb{Z}$. Then P^\vee is a sublattice of \widehat{P}^\vee . The associated DAHA is defined as follows.

Definition 6.1. (See [9,5].) Double Affine Hecke Algebra of type \widehat{R}^\vee is the algebra $\mathcal{H}_t^{\widehat{R}^\vee}$ generated by the elements T_0, T_1, \dots, T_n , by the set of pairwise commuting elements $\{X^\mu\}_{\mu \in \widehat{P}}$ and by the group $\Pi = \{\pi_r \mid r \in \mathcal{O}\}$ with the relations (2.4)–(2.8), (2.11) (with \widehat{P}^\vee instead of \widehat{P}) and

$$T_i^{-1} X^\mu T_i^{-1} = X^{\mu - \alpha_i^\vee}, \quad \text{if } (\mu, \alpha_i) = 1, \mu \in \widehat{P}^\vee; \quad (6.1)$$

$$T_i X^\mu = X^\mu T_i, \quad \text{if } (\mu, \alpha_i) = 0, \mu \in \widehat{P}^\vee; \quad (6.2)$$

where $i = 0, 1, \dots, n$.

Consider the space $\check{\mathcal{F}}$ of functions $f \in C^\infty(V)^{2\pi i Q}$ holomorphic in a neighbourhood U_f of $\check{Z} = \bigcup_{\alpha \in \widehat{R}^\vee} \{x \in V \mid \alpha(x) \in 2\pi i \mathbb{Z}\}$, where $C^\infty(V)^{2\pi i Q}$ is the space of smooth functions $f : V \rightarrow \mathbb{C}$ such that $f(x + 2\pi i \alpha_j) = f(x)$, $\forall j = 1, \dots, n$. The representation of DAHA $\mathcal{H}_t^{\widehat{R}^\vee}$ in this space is defined by the formulae

$$X^\mu = e^{\mu(x)}, \quad \pi_r = \pi_r, \quad T_i = \frac{t_i - t_i^{-1}}{1 - e^{-\alpha_i^\vee(x)}} + \frac{t_i^{-1} - t_i e^{-\alpha_i^\vee(x)}}{1 - e^{-\alpha_i^\vee(x)}} s_i, \quad (6.3)$$

where $\mu \in \widehat{P}^\vee, r \in \mathcal{O}, i = 0, 1, \dots, n$, so

$$T_i f(x) = t_i f(x) + (t_i^{-1} - t_i e^{-\alpha_i^\vee(x)}) \tilde{g}(x) \quad (6.4)$$

where $f \in \check{\mathcal{F}}$ and $\tilde{g}(x) = \frac{f(s_i x) - f(x)}{1 - e^{-\alpha_i^\vee(x)}} \in \check{\mathcal{F}}$.

Let $m_\lambda(Y)$ be the elements (2.22) of $\mathcal{H}_t^{\widehat{R}^\vee}$ in the representation (6.3), where Y^λ are defined by the formulae (2.12), (2.14). Their restrictions to $\check{\mathcal{F}}^W$ are difference operators $M_\lambda = \text{Res}(m_\lambda(Y))$ spanning the commutative algebra \mathfrak{R} . The algebra \mathfrak{R} is generated by the algebraically independent operators $M_{-b_1}, \dots, M_{-b_n}$, where b_i are the fundamental coweights for R . This is the Macdonald–Ruijsenaars system corresponding to the affine root system \widehat{R}^\vee . In particular, it always contains the Macdonald operator

$$M_{-\theta^\vee} = \sum_{v \in W\theta^\vee} \left(\prod_{\substack{\alpha \in R \\ (\alpha, v) > 0}} \frac{t_{\alpha^\vee} e^{(\alpha^\vee, x)} - t_{\alpha^\vee}^{-1}}{e^{(\alpha^\vee, x)} - 1} \right) \frac{t_0 e^{(v, x) - \hbar(\theta^\vee, \theta^\vee)/2} - t_0^{-1}}{e^{(v, x) - \hbar(\theta^\vee, \theta^\vee)/2} - 1} (\tau(v) - 1) \\ + m_{-\theta^\vee, t}. \quad (6.5)$$

Let $S_0 \subset \Delta$ and let $\eta : \widehat{R}^\vee \rightarrow \mathbb{C}$ be a \widehat{W} -invariant function (it is sufficient to define it on the subsystem $R^\vee \subset \widehat{R}^\vee$). Define the corresponding ideal $I_{\mathcal{D}} \subset \mathcal{F}$ as follows

$$W_+ = \{w \in W \mid wS_0 \subset R_+\} \subset W, \quad (6.6)$$

$$\mathcal{D}_0 = H((S_0^\vee)') = \{x \in V \mid (\alpha^\vee, x) = \eta_{\alpha^\vee}, \forall \alpha \in S_0\}, \quad (6.7)$$

$$\mathcal{D} = W_+ \mathcal{D}_0, \quad I_{\mathcal{D}} = \{f \in \mathcal{F} \mid f(x) = 0, \forall x \in \mathcal{D}\}. \quad (6.8)$$

Let $R_0 = \bigsqcup_{\ell=1}^m R_\ell$ and $S_0 = \bigsqcup_{\ell=1}^m S_\ell$ be the decompositions into irreducible components, where $R_0 = \langle S_0 \rangle \cap R$. Let $\theta_\ell \in R_\ell$ be the maximal root with respect to the basis S_ℓ . Let $S_\ell = \{\alpha_j^\ell \mid j = 1, \dots, k_\ell\}$, $\theta_\ell = \sum_{j=1}^{k_\ell} n_j^\ell \alpha_j^\ell$ and $\eta_j^\ell = \eta_{(\alpha_j^\ell)^\vee}$. Define the corresponding generalized Coxeter numbers as

$$\check{h}_\eta^\ell = \frac{(\theta_\ell, \theta_\ell)}{2} \eta_{\theta_\ell^\vee} + \sum_{j=1}^{k_\ell} \frac{(\alpha_j^\ell, \alpha_j^\ell)}{2} n_j^\ell \eta_j^\ell. \quad (6.9)$$

Theorem 6.1. Suppose the subset $S_0 \subset \Delta$ and the invariant functions η, t satisfy $\check{h}_\eta^\ell = \check{h}$ for all $\ell = 1, \dots, m$ and $t_{\alpha^\vee}^2 = e^{\eta_{\alpha^\vee}}$ for all $\alpha \in R$. Then the ideal $I_{\mathcal{D}} \subset \check{\mathcal{F}}$ is invariant under the action of $\mathcal{H}_t^{\hat{R}^\vee}$.

Proof. First we define the W -invariant function $\eta' : R \rightarrow \mathbb{C}$ by the formula $\eta'_\alpha = \frac{(\alpha, \alpha)}{2} \eta_{\alpha^\vee}$. Note that one can represent (6.7) as

$$\mathcal{D}_0 = H_{\eta'}(S_0) = \{x \in V \mid (\alpha, x) = \eta'_\alpha, \forall \alpha \in S_0\}. \quad (6.10)$$

By virtue of (6.10) the ideal (6.8) for the function $\eta : R^\vee \rightarrow \mathbb{C}$ coincides with the ideal (3.12) for the function η' , moreover the conditions $\check{h}_\eta^\ell = \check{h}$ are equivalent to $h_{\eta'}^\ell = \check{h}$ where $h_{\eta'}^\ell$ is the generalized Coxeter number for the function η' . Hence the invariance under the action of π_r follows from Proposition 3.7.

Let $f \in I_{\mathcal{D}}$ and $x \in w\mathcal{D}_0$ for some $w \in W_+$. Consider first the invariance under T_i with $i = 1, \dots, n$. In the case $\alpha_i \in wS_0$ it follows from $f(x) = 0$ and $(t_i^{-1} - t_i e^{-\alpha_i^\vee(x)}) = 0$. In the case $\alpha_i \notin wS_0$ we have $f(x) = f(s_i x) = 0$ and the plane $w\mathcal{D}_0 = H_{\eta'}(wS_0)$ does not lie in any plane $H([\alpha_i^\vee, 2\pi i k]) = H([\alpha_i, \pi i k(\alpha_i, \alpha_i)])$ due to Lemma 3.4 (see the proof of Proposition 3.5 for detail).

Consider now the invariance under T_0 . If $w\theta_\ell = \theta$ for some ℓ then $\alpha_0^\vee(x) = \eta_{\theta^\vee}$ and $t_0^{-1} - t_0 e^{-\alpha_0^\vee(x)} = 0$; so that $T_0 f(x) = 0$. If $w\theta_\ell \neq \theta$ for all $\ell = 1, \dots, m$ then the plane $s_0\mathcal{D}_0 = s_0 H_{\eta'}(wS_0)$ contains in \mathcal{D} due to Lemma 3.9 and the condition $h_{\eta'}^\ell = \check{h}$. Hence $f(x) = f(s_0 x) = 0$ and $T_0 f(x) = 0$ for $x \in w\mathcal{D}_0$ such that $e^{\alpha_0^\vee(x)} \neq 1$. Thus we are left to prove that $w\mathcal{D}_0 = H_{\eta'}(wS_0)$ does not lie in the affine hyperplane $H(\alpha_0^\vee + [0, 2\pi i k]) = H(\alpha_0 + [0, \pi i k(\theta, \theta)])$. This is done in the same way as in the proof of Proposition 3.10. \square

Let \overline{M}_λ be the restriction of the difference operator M_λ to the plane \mathcal{D}_0 defined as in Section 4.1. Repeating the reasoning of Sections 4.1 and 4.2 we conclude that the operators \overline{M}_λ span the commutative algebra $\overline{\mathfrak{M}}$ containing $\bar{n} = \dim \mathcal{D}_0$ algebraically independent elements. This is the generalized Macdonald–Ruijsenaars system for the \hat{R}^\vee case.

7. Concluding remarks

We considered submodules in the polynomial representation of DAHA given by vanishing conditions and derived the corresponding generalized Macdonald–Ruijsenaars operators and their quantum integrals acting in the quotient module. In some particular cases the eigenfunctions of these operators can be given by Baker–Akhiezer functions [2,3,12] but more generally the understanding of eigenfunctions is a major challenge in the area. It follows from our construction that the restriction of a Weyl-invariant eigenfunction of the higher Macdonald operators M_λ

onto the plane \mathcal{D}_0 is an eigenfunction of the corresponding operators \overline{M}_λ . However the question of finding the solutions is non-trivial as the specializations of the parameters so that the ideals $I_{\mathcal{D}}$ are invariant are such that the Macdonald polynomials do not exist in general at these specializations. We also note that more generally, the non-symmetric generalized eigenfunctions for the Cherednik–Dunkl operators at the special parameters are studied in [10]. Further comparison with [10] may be important for extracting the information on the eigenfunctions of the operators \overline{M}_λ . At generic values of the parameters for the restricted operators the eigenfunctions might have different structure and description. Thus in type A a complete set of eigenfunctions (so-called super Macdonald polynomials) is constructed in [29] at generic η, \hbar , these polynomials are symmetric in each set of variables x_i, y_j and span a subspace in the space of functions of x_i, y_j with simple description. It would be interesting to obtain Pieri formulae for the super Macdonald polynomials, and Theorem 4.5 might be useful.

Another important question is on the composition series of the polynomial representation. In the case of $R = A_N$ it is given in terms of the ideals defined by vanishing (“wheel”) conditions [18,11]. It would be interesting to see if the ideals $I_{\mathcal{D}}$ can be used to define the composition factors in some other cases. Further, the vanishing conditions of the submodules in the polynomial representation for the rational Cherednik algebras were helpful to establish unitarity of certain submodules [14]. It would be interesting to see if analogous approach can be applied for the case of DAHA.

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Appendix A. Other affine root systems

The Macdonald–Ruijsenaars systems and their generalized versions can be defined starting from a general affine root system. The cases of affine root systems not considered in the main part of the paper lead to particular cases of the operators \overline{M}_λ from Section 5. Nevertheless we present these cases for completeness.

Let $\mathbb{S} \subset \widehat{V}$ be an irreducible affine root system [23,25] and let $\widetilde{W} = \widetilde{W}(\mathbb{S})$ be the group generated by $\{s_\alpha \mid \alpha \in \mathbb{S}\}$ – affine Weyl group for the affine root system \mathbb{S} . Let $\Delta_1 = \{a_0, a_1, \dots, a_n\}$ be a basis of \mathbb{S} and let

$$\bar{a}_i = \begin{cases} a_i, & \text{if } 2a_i \notin \mathbb{S}, \\ 2a_i, & \text{if } 2a_i \in \mathbb{S}, \end{cases} \quad \Delta_2 = \{\bar{a}_0, \bar{a}_1, \dots, \bar{a}_n\}. \quad (\text{A.1})$$

The sets $\mathbb{S}_1 = \{\alpha \in \mathbb{S} \mid \alpha/2 \notin \mathbb{S}\}$ and $\mathbb{S}_2 = \{\alpha \in \mathbb{S} \mid 2\alpha \notin \mathbb{S}\}$ are reduced affine root subsystems in \widehat{V} and Δ_1 and Δ_2 are bases of \mathbb{S}_1 and \mathbb{S}_2 respectively. The group \widetilde{W} is generated by the affine simple reflections $s_i = s_{a_i} = s_{\bar{a}_i}$.

Let $t : \mathbb{S}_2 \rightarrow \mathbb{C} \setminus \{0\}$ and $u : \mathbb{S}_2 \rightarrow \mathbb{C} \setminus \{0\}$ be \widetilde{W} -invariant functions such that $u_\alpha = u(\alpha) = 1$ for all $\alpha \in \mathbb{S}_1 \cap \mathbb{S}_2$. Let $t_i = t_{\bar{a}_i}$, $u_i = u_{\bar{a}_i}$, $i = 0, 1, \dots, n$. The functions t and u are completely

determined by these values and the restriction condition on u can be reformulated as $u_i = 1$ if $a_i = \bar{a}_i$. Consider the operators

$$T_i = t_i + t_i^{-1} \frac{(1 - t_i u_i e^{-\bar{a}_i(x)/2})(1 + t_i u_i^{-1} e^{-\bar{a}_i(x)/2})}{1 - e^{-\bar{a}_i(x)}} (s_i - 1), \quad i = 0, 1, \dots, n. \quad (\text{A.2})$$

These operators preserve the space \mathcal{F} of smooth functions $f \in C^\infty(V)$ satisfying $f(x) = f(x + 2\pi i(\bar{a}_j^{(1)})^\vee)$ for all $j = 0, 1, \dots, n$ and holomorphic in a neighbourhood U_f of the set $\bigcup_{\alpha \in \mathbb{S}} \{x \in V \mid \alpha(x) \in 2\pi i\mathbb{Z}\}$.

Proposition A.1. *The operators (A.2) satisfy the affine Hecke algebra relations (2.6) and (2.7) where m_{ij} is the order of the element $s_i s_j \in \tilde{W}$.*

The proposition follows from the established cases $\mathbb{S} = (C_n^\vee, C_n)$ and $\mathbb{S} = \hat{R}$ where R is a reduced root system.

Now consider the affine root systems not covered in the previous sections. These are $\mathbb{S} = (B_n, B_n^\vee)$, $\tilde{B}C_n$, (BC_n, C_n) and (C_n^\vee, BC_n) . In the above notations we have $\mathbb{S}_1 = \widehat{B}_n$, $\mathbb{S}_2 = \widehat{B}_n^\vee$ for $\mathbb{S} = (B_n, B_n^\vee)$, $\mathbb{S}_1 = \mathbb{S}_2 = \tilde{B}C_n$ for $\mathbb{S} = \tilde{B}C_n$; $\mathbb{S}_1 = \tilde{B}C_n$, $\mathbb{S}_2 = \widehat{C}_n$ for $\mathbb{S} = (BC_n, C_n)$; $\mathbb{S}_1 = \widehat{C}_n^\vee$, $\mathbb{S}_2 = \tilde{B}C_n$ for $\mathbb{S} = (C_n^\vee, BC_n)$. We consider the standard realizations of these affine root systems, so in particular, $\mathbb{S} \supset D_n = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\}$. Define the lattice $L_Y = \mathbb{Z}^n = P_{B_n}^\vee = Q_{C_n}^\vee = P_{D_n}^\vee$, where $P_{B_n}^\vee$, $Q_{C_n}^\vee$ and $P_{D_n}^\vee$ are the coweight lattice for $R = B_n$, the coroot lattice for $R = C_n$ and the coweight lattice for $R = D_n$ respectively. Let $H_{\mathbb{S}}^{L_Y}$ be the affine Hecke algebra generated by the operators $T_{\hat{w}} = T_{\hat{w}}^{\mathbb{S}}$ with $\hat{w} \in W \ltimes \tau(L_Y)$ defined by the formulae (2.12), (A.2). So we consider ‘extended’ affine Hecke algebra for $\mathbb{S} = (B_n, B_n^\vee)$ and ‘non-extended’ for other cases as the group $W \ltimes \tau(L_Y)$ is the extended affine Weyl group $\tilde{W} = \Pi \ltimes \tilde{W}(\mathbb{S})$ for $\mathbb{S} = (B_n, B_n^\vee)$ and it is the usual affine Weyl group $\tilde{W} = \tilde{W}(\mathbb{S})$ for other \mathbb{S} . We will also consider the ‘non-extended’ affine Hecke algebras $H_{\mathbb{S}}^{L_Y}$ for $\mathbb{S} = \widehat{C}_n$ and \widehat{C}_n^\vee , where L_Y is as above. The Cherednik–Dunkl operators $Y^\lambda = Y_{\mathbb{S}}^\lambda$, $\lambda \in L_Y$, are defined by the formula $Y^{\mu-\nu} = T_{\tau(\mu)} T_{\tau(\nu)}^{-1}$, where $\mu, \nu \in P_+^\vee \cap L_Y$. These operators form the commutative subalgebra $\mathbb{C}[Y_{\mathbb{S}}^{L_Y}] \subset H_{\mathbb{S}}^{L_Y}$.

The elements $m_\lambda(Y) = \sum_{v \in W_\lambda} Y^v$ preserve the space of invariants \mathcal{F}^W , their restrictions $M_\lambda = m_\lambda|_{\mathcal{F}^W}$ form commutative algebra \mathfrak{R}^{L_Y} of difference operators. The simplest operators in this algebra has the form

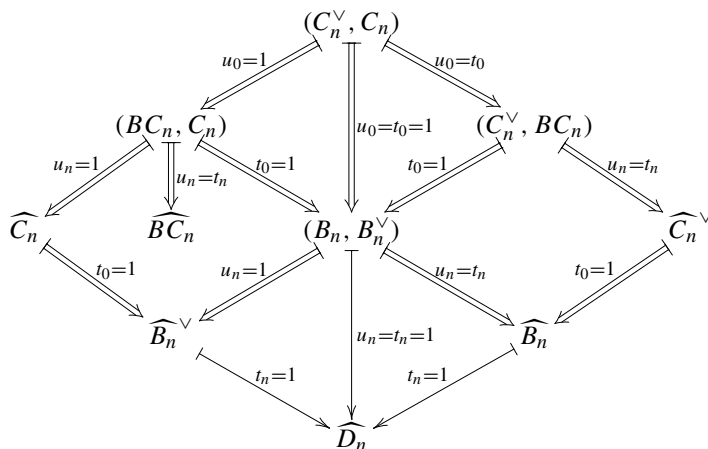
$$M_{-\nu} = \frac{1}{|W_{-\nu}|} \sum_{w \in W} g^{-\nu}(w^{-1}x)(\tau(-w\nu) - 1) + \text{const}, \quad (\text{A.3})$$

where ν is a minuscule coweight b_r or $\nu = \theta^\vee$, and

$$g^{-\nu}(x) = \prod_{\alpha \in \mathbb{S}_2(-\nu)} \frac{(1 - t_\alpha u_\alpha e^{\alpha(x)/2})(1 + t_\alpha u_\alpha^{-1} e^{\alpha(x)/2})}{t_\alpha (1 - e^{\alpha(x)})}, \quad (\text{A.4})$$

where $\mathbb{S}_2(-\nu) = \mathbb{S}_2^- \cap (\tau(-\nu)\mathbb{S}_2^+)$, $\mathbb{S}_2^+ = -\mathbb{S}_2^- = \{\alpha \in \mathbb{S}_2 \mid \alpha(x) > 0 \text{ whenever } \bar{a}_i(x) > 0, \forall i = 0, 1, \dots, n\}$.

The following diagram explains how one gets the subalgebras $\mathbb{C}[Y_{\mathbb{S}}^{L_Y}] \subset H_{\mathbb{S}}^{L_Y}$ by specializing the parameters starting from the subalgebra generated by the Cherednik–Dunkl operators in the (C_n^\vee, C_n) DAHA. We also include in the diagram the affine Hecke algebras parts of DAHA of \hat{R} and \hat{R}^\vee type where $R = B_n, D_n$.



The double arrows in the diagram mean the isomorphism of the corresponding Cherednik–Dunkl subalgebras under the specialization and the single arrows mean the embedding of these subalgebras. The parameters t_0, t_n, u_0, u_n are the parameters of the operators (A.2) for $\mathbb{S} = (C_n^{\vee}, C_n)$. The validity of these isomorphisms essentially follows from the following two lemmas.

Lemma A.2. *If $t_0 = u_0 = 1$ then the affine Hecke algebra $H_{(C_n^{\vee}, C_n)}^{Q^{\vee}}$ coincides with the affine Hecke algebra $H_{(B_n, B_n^{\vee})}^{P^{\vee}}$. More exactly, under the specialization the generators correspond as*

$$T_i^{(C_n^{\vee}, C_n)} = T_i^{(B_n, B_n^{\vee})}, \quad T_0^{(C_n^{\vee}, C_n)} = \pi_1^{(B_n, B_n^{\vee})}, \quad (\text{A.5})$$

where $i = 1, \dots, n$, and the Cherednik–Dunkl operators correspond as $Y_{(C_n^{\vee}, C_n)}^{\lambda} = Y_{(B_n, B_n^{\vee})}^{\lambda}$, $\forall \lambda \in L_Y$.

Proof. The affine Hecke algebra $H_{(C_n^{\vee}, C_n)}^{Q^{\vee}}$ is generated by $T_0^{(C_n^{\vee}, C_n)}, T_1^{(C_n^{\vee}, C_n)}, \dots, T_n^{(C_n^{\vee}, C_n)}$ while the algebra $H_{(B_n, B_n^{\vee})}^{P^{\vee}}$ is generated by $\pi_1^{(B_n, B_n^{\vee})}, T_1^{(B_n, B_n^{\vee})}, \dots, T_n^{(B_n, B_n^{\vee})}$. The operators $T_i^{(C_n^{\vee}, C_n)}$ and $T_i^{(B_n, B_n^{\vee})}$ coincide for each $i = 1, \dots, n$ (at any parameters) so denote them by T_i . The operator $T_0^{(C_n^{\vee}, C_n)}$ at the specialization of parameters becomes $s_0^{(C_n^{\vee}, C_n)} = \tau(e_1)s_{e_1} = \pi_1^{(B_n, B_n^{\vee})}$. Thus we are left to prove the formula $Y_{(C_n^{\vee}, C_n)}^{\lambda} = Y_{(B_n, B_n^{\vee})}^{\lambda}$ at this specialization. It is enough to check this formula for $\lambda = e_i$, $i = 1, \dots, n$. From the definition one yields $Y_{(C_n^{\vee}, C_n)}^{e_1} = T_0^{(C_n^{\vee}, C_n)} T_{s_{e_1}}^{(C_n^{\vee}, C_n)}$ and $Y_{(B_n, B_n^{\vee})}^{e_1} = \pi_1^{(B_n, B_n^{\vee})} T_{s_{e_1}}^{(B_n, B_n^{\vee})}$. Then for $i = 2, \dots, n$ the equality $Y_{(C_n^{\vee}, C_n)}^{e_i} = Y_{(B_n, B_n^{\vee})}^{e_i}$ follows by induction from the formula $Y_{\mathbb{S}}^{e_i} = T_{i-1}^{-1} Y_{\mathbb{S}}^{e_{i-1}} T_{i-1}^{-1}$ valid for both $\mathbb{S} = (C_n^{\vee}, C_n)$ and $\mathbb{S} = (B_n, B_n^{\vee})$. \square

Consider now $\mathbb{S} = \widehat{D_n}$, $n \geq 2$. The simple roots of D_n are $\alpha_i = e_i - e_{i+1}$ for $i = 1, \dots, n-1$ and $\alpha_n = e_{n-1} + e_n$. Denote by $\pi_r^{D_n}, T_i^{D_n}$ the corresponding elements of the algebra $H_{D_n}^{P^{\vee}}$ where $r \in \mathcal{O} = \{0, 1, n-1, n\}$, $i = 0, 1, \dots, n$.

Note that if $t_n = 1$ then the affine Hecke algebra $H_{B_n}^{P^{\vee}}$ in the representation (2.19) coincides with the affine Hecke algebra $H_{B_n^{\vee}}^{P^{\vee}}$ in the representation (6.3). Indeed, the algebras $H_{B_n}^{P^{\vee}}, H_{B_n^{\vee}}^{P^{\vee}}$

have the only different generator T_n which becomes $T_n = s_{e_n}$ at $t_n = 1$. Since the corresponding operators π_1, T_i of the algebras $H_{B_n}^{P^\vee}$ and $H_{B_n^\vee}^{P^\vee}$ coincide at this specialization we denote them by $\pi_1^{B_n}, T_i^{B_n}$ where $i = 0, \dots, n$.

Lemma A.3. *At the specialization $t_n = 1$ one has*

$$Y_{B_n}^\lambda = Y_{B_n^\vee}^\lambda = Y_{D_n}^\lambda \quad (\text{A.6})$$

for all $\lambda \in L_Y$. That is the corresponding algebras are embedded at this specialization: $\mathbb{C}[Y_{B_n}^{P^\vee}] = \mathbb{C}[Y_{B_n^\vee}^{P^\vee}] \subset \mathbb{C}[Y_{D_n}^{P^\vee}]$.

Proof. Note first that the operators $T_i^{B_n}$ and $T_i^{D_n}$ coincide for $i = 1, \dots, n-1$ (at any parameters) and we denote them by T_i . Since $T_n^{B_n} = s_{e_n}$ (at the specialization $t_n = 1$) we obtain

$$\pi_1^{D_n} = \tau(e_1)s_{e_1}s_{e_n} = \pi_1^{B_n}T_n^{B_n}, \quad T_n^{D_n} = s_nT_{n-1}s_n = (T_n^{B_n})^{-1}T_{n-1}T_n^{B_n}. \quad (\text{A.7})$$

Comparing the expressions

$$Y_{B_n}^{e_1} = \pi_1^{B_n}T_1 \dots T_{n-2}T_{n-1}T_n^{B_n}T_{n-1} \dots T_1, \quad (\text{A.8})$$

$$Y_{D_n}^{e_1} = \pi_1^{D_n}T_1 \dots T_{n-2}T_n^{D_n}T_{n-1} \dots T_1 \quad (\text{A.9})$$

we derive $Y_{B_n}^{e_1} = Y_{D_n}^{e_1}$. Due to the formula $Y_R^{e_i} = T_{i-1}^{-1}Y_R^{e_{i-1}}T_{i-1}^{-1}$ valid for both $R = B_n$ and $R = D_n$ we establish the equality (A.6) for all $\lambda = e_i$, where $i = 1, \dots, n$, and, hence, for all $\lambda \in L_Y$. \square

It follows that the operators M_λ for the affine root system \mathbb{S} coincide with the operators M_λ from Section 5 at the corresponding specialization. In particular the specializations of the Koornwinder operator (5.8) give the operators M_{-e_1} determined by (A.3). The specializations of the generalized Koornwinder operator (5.12) give the generalized Macdonald–Ruijsenaars operators \overline{M}_{-e_1} for the affine root system \mathbb{S} . Each specialization of the operators \overline{M}_λ from Section 5 gives commutative algebra containing the operator \overline{M}_{-e_1} . These algebras were constructed in Sections 4, 6 for $\mathbb{S} = \widehat{B}_n, \widehat{D}_n, \widehat{B}_n^\vee$. For other cases the invariant ideals and these operators can also be defined using the specialization of the invariant ideals for $\mathbb{S} = (C_n^\vee, C_n)$.

For example consider the case $\mathbb{S} = (B_n, B_n^\vee)$, so $\mathbb{S}_2 = \widehat{B}_n^\vee$. Let $\eta: B_n^\vee \rightarrow \mathbb{C}$ be a W -invariant function and let S_0 be a subset of the set of simple roots of B_n . Define the ideal $I_{\mathcal{D}}$ by the formulae (6.7) and (6.8). Let $S_0 = \bigsqcup_{\ell=1}^m S_\ell$ be the decomposition to the irreducible components $S_1, \dots, S_m \subset S_0$. Let \check{h}_η^ℓ be the corresponding generalized Coxeter number defined by (6.9). Let $\check{\mathcal{F}}$ be the space defined in Section 6 for $R = B_n$. Then the following statement takes place.

Theorem A.4. *Suppose that $t_1^2 = e^{\eta_1}$ if $n > 1$, $\epsilon_n t_n u_n^{\epsilon_n} = e^{\eta_n/2}$ for some $\epsilon_n = \pm 1$ and $\check{h}_\eta^\ell = \hbar$ for all $\ell = 1, \dots, m$. Then the ideal $I_{\mathcal{D}} \subset \check{\mathcal{F}}$ is invariant under the actions of the operators (A.2) and π_1 . The corresponding generalized Macdonald–Ruijsenaars operator \overline{M}_{-e_1} can be obtained by the specialization of the parameters in the generalized Koornwinder operator (5.12).*

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